Centre for Distance and Online Education

Punjabi University, Patiala

Class : B.A. I (Mathematics) **Semester : I**

Paper : MTHB1102T **Unit-II**

(Algebra and Trigonometry)

Medium : English

Lesson No.

Department website : www.pbidde.org

MTHB1102T: ALGEBRA AND TRIGONOMETRY

Course Outcomes:	
CO1	To understand D'Moivre's theorem, application of D'Moivre's theorem.
CO2	To know about exponential, logarithmic, direct and inverse circular and hyperbolic functions of a complex variable.
CO3	To understand Summation of series including Gregory Series.
CO4	To know Hermitian and skew-hermitian matrices,linear dependence of row and column vectors.
CO5	To understand Eigen-values, eigen-vectors and characteristic equation of a matrix.

For Regular Students / Students of Centre for Distance and Online Education
Maximum Marks: 50 Marks

Maximum Time: 3 Hrs.
For Regular students:6Lectures of 45minutes/week

External Marks: 35
Internal Assessment: 15
Pass Percentage: 35%
For Private Students
Maximum Marks: 50 Marks

INSTRUCTIONS FOR THE PAPER-SETTER

The question paper will consist of three sections A, B and C. Sections A and B will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eleven short answer type questions covering the entire syllabus uniformly. Each question in Sections A and B will be of 06 marks and Section C will be of 11 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

SECTION-A

D'Moivre's theorem, application of D'Moivre's theorem including primitive nth root of unity. Expansions of $\sin n\theta$, $\cos n\theta$, $\sin^n\theta$, $\cos^n\theta$ $(n \in N)$. The exponential, logarithmic, direct and inverse circular and hyperbolic functions of a complex variable. Summation of series including Gregory Series.

SECTION-B

Hermitian and skew-hermitian matrices, linear dependence of row and column vectors, row rank, column rank and rank of a matrix and their equivalence. Theorems on consistency of a system of linear equations (both homogeneous and non-homogeneous). Eigen-values, eigen-vectors and characteristic equation of a matrix, Cayley-Hamilton theorem and its use in finding inverse of a matrix. Diagonalization.

Head,
Department of Mathematic
Punjabi University, Patiala

REFERENCES:

1. K.B. Datta : Matrix and Linear Algebra, Prentice Hall of India Pvt. Ltd., New Delhi, 2000.

2. S. R. Knight and H.S. Hall : Higher Algebra, H.M. Publications, 1994.

3. R.S. Verma and K.S. Shukla : Text Book on Trigonometry, Pothishala Pvt. Ltd., Allahabad.

4. Shanti Narayan and P.K. Mittal : A Text Book of Matrices, S. Chand & Co., New Delhi, Revised Edition, 2007.

Head,
Department of Mathematics
Punjabi University, Patiala

UPDATED ON APRIL, 2024

B.A. PART– I **MTHB1102T**

(SEMESTER - I) **ALGEBRA AND TRIGONOMETRY**

LESSON NO. 2.1 **AUTHOR : DR. CHANCHAL**

RANK OF A MATRIX

2.1.1 Objectives

The prime objective of this lesson is

• To understand the concept of rank of a matrix

• To study elementary transformations

• To determine rank and inverse of a matrix using elementary transformations

2.1.2 Introduction

To understand the concept of rank of a matrix, firstly we take a look at the various types of matricas the have studied in our presbes class.

(I) Transpose of a matrix : The matrix obtained from a given matrix A, by interchanging its rows and columns, is called the transpose of A and is generally denoted by A' or A^t or A^T.

Thus if $A = \left[a_{ij}\right]$ then (j, i)th element of A' is equal to (i, j) element of A

For example : If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$

Remarks :

(1) (A')' = A (2) (A + B)' = A' + B'

(3) (k A)' = k A', l being a complex number. (4) (AB)' = B'A'

(II) Symmetric and Skew-Symmetric Matrices

1. Any square matrix A = $[a_{ij}]$ is said to be a symmetric matrix if $a_{ij} = a_{ji}$ i.e., (i, j)th element of A is the same as the (j, i)th element of A. If we take the transpose of a symmetric matrix A, it is the same as A.

For example : $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

2. Any square matrix $A = \left[a_{ij}\right]$ is said to be a skew-symmetric matrix if $a_{ij} = -a_{ji}$ i.e. (i, j)th element is the same as the negative of the (j, i) element.

∴ for a skew-symmetric matrix A, $a_{ij} = -a_{ji}$

If we put j = i, we get $a_{ij} = -a_{ji}$ or $a_{ii} = 0$ i.e, every diagonal element of A is zero.

Examples of skew-symmetric matrix are

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

(III) Conjugate and Tranjugate of a Matrix

The Matrix obtained by replacing the elements by A by its complex conjugates, is called the conjugate of A and is generally denoted by A.

Thus, if $A = \left[a_{ij}\right], \bar{A} = \left[\bar{a}_{ij}\right]$ where denotes the conjugate of a_{ij}

For example : if $A = \begin{bmatrix} 2+3i & 7-5i & 6+i \\ 5 & 2+3i & 1-2i \\ -3-5i & 0 & 2-5i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 2-3i & 7+5i & 6-i \\ 5 & 2-3i & 1+2i \\ -3-5i & 0 & 2+5i \end{bmatrix}$

If all the element of A are real, then $\bar{A} = A$.

Note $\overline{(\bar{A})} = A$

Tranjugate of Matrix

The conjugate of the transpose of a matrix A is called tranjugate of A and is denoted by A^θ. Thus $A^\theta = \overline{(A')}$

Clearly $\overline{(A')} = (\bar{A})'$

$\therefore \quad A^\theta = (\bar{A})'$

For example : if

$$A = \begin{bmatrix} 2+3i & 6-i & 5+2i \\ 3 & 2 & -1+5i \\ 0 & 7-3i & -5+6i \end{bmatrix}$$

then $A^\theta = \begin{bmatrix} 2-5i & 3 & 0 \\ 6+i & 2 & 7+3i \\ 5-2i & -1-5i & -5-6i \end{bmatrix}$

Note : $(A^\theta)^\theta = A$.

(IV) Hermitian and Skew-Hermitian Matrices

(1) A square matrix $A = [a_{ij}]$ is said to be hermitian if $a_{ij} = \bar{a}_{ij}$ i.e., (i, j)th element is the conjugate of the (j, i)th element.

Now, $d\, a_{ij} = \bar{a}_{ij} \therefore a_{ii} = \bar{a}_{ii}$ i.e., the conjugate of any diagonal element is the same element.

$\therefore$ every diagonal element must be real.

For example :

$$\begin{bmatrix} 2 & 5-6i & 3-4i \\ 5+6i & 0 & 1-2i \\ 3+4i & 1+2i & 7 \end{bmatrix}, \begin{bmatrix} 0 & a+ib & c+id \\ a-ib & 1 & m+in \\ c-id & m-in & p \end{bmatrix}$$

(2) A square matrix $A = [a_{ij}]$ is said to be Skew-hermitian if $a_{ij} = -\bar{a}_{ji}$ i.e., (i, j)the element is the negative conjugate of (j, i) element.

Again as $a_{ij} = -\bar{a}_{ji} \therefore a_{ii} = -\bar{a}_{ii}$ i.e, $a_{ii} + \bar{a}_{ii} = 0$.

$\therefore$ every diagonal element must be either zero or a purely imaginary number.

For example :

$$\begin{bmatrix} 4i & 4-3i & 6+5i \\ -4-3i & 0 & 2+7i \\ -6+5i & -2+7i & -9i \end{bmatrix}, \begin{bmatrix} 5i & 3-7i \\ -3-7i & 9i \end{bmatrix}$$

(v) Orthogonal Matrix

A square matrix P over the field of reals is said to be orthogonal if and only if P'P = I.
Now, if P is orthogonal, then P'P = I = PP'.

$\Rightarrow |P'P| = |I| \quad \Rightarrow |P'||P| = I$

$\Rightarrow |P||P| = I \quad \Rightarrow |P|^2 = I$

$\Rightarrow |P| = \pm I \quad \Rightarrow |P| \neq 0$

$\Rightarrow$ P is invertible

$\therefore$ If P is orthogonal, then P is invertible.

Also P'P = I $\Rightarrow$ $P' = P^{-1}$

$\Rightarrow$ $PP' = PP^{-1}$ $\Rightarrow$ PP' = I

$\therefore$ P is orthgonal iff P'P = PP' = I i.e, iff $P' = P^{-1}$.

For example :

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(vi) Unitary Matrix

A square matrix P over the fied of complex numbers is said to be unitary if and only if $P^{\theta}P$ I = PP^{θ}.

Now, if P is unitary, then $P^\theta P = I$

$$\Rightarrow |P^\theta P| = |I| \Rightarrow |P^\theta||P| = 1 \quad \Rightarrow |P||P| = 1 \quad \Rightarrow \left[\therefore |P^\theta| = |P|\right]$$

$$\Rightarrow |P|^2 = I \Rightarrow |P| \neq 0$$

$\Rightarrow$ P is invertible

$\therefore$ if P is unitary, then P is invertible.

Also $P^\theta P \Rightarrow I \Rightarrow P^\theta = P^{-1} \Rightarrow PP^\theta = PP^{-1} \Rightarrow PP^\theta = I$

$\therefore |P| = \pm 1$

$\Rightarrow$ absolute value of a determinant of a unitary matrix is 1.

(vii) Similar Matrices

Let A and B be square matrices of order n over a field. Then A is said to be similar to B over F if and only if there exists an n-rowed invertible matrix C over F such that

$AC = CB$ i.e. $B = C^{-1} AC$ or $A = CB^{-1}$.

2.1.3 Rank of a Matrix

Definition : A number r is said to be rank of a non-zero matrix A if

(i) there exists at least one minor of order r of A which does not vanish, and

(ii) every minor of order (r + 1), if any, vanishes

The rank of a matrix A is denoted by $\rho(A)$.

$\therefore$ We have $\rho(A) = r$.

In other words, the rank of a non-zero matrix is the largest order of any non-vanishing minor of the matrix.

Remarks : (i) the rank of a zero matrix is zero i.e., $\rho(O) = 0$ where O is a zero matrix.

(ii) the rank of a non-singular matrix of order n is n,

(iii) $\rho(A) \leq r$, if every minor of order (r + 1) vanishes,

(iv) $\rho(A) \geq r$, if there is a minor of order r whch does not vanish.

Some Important Results :-

Result 1: Prove that the rank of the transpose of a matrix A is the same as that of the original matrix A.

Proof . If $A = O$, then $A' = O$

$\therefore \quad \rho(A) = 0$ and $\rho(A') = 0$

$\Rightarrow \quad \rho(A') = \rho(A)$

$\therefore$ result is true in the case in which A is a zero matrix.

Now we discuss the case when $A \neq O$.

Let r be the rank of the matrix $A = [a_{ij}]$ where A is of type $m \times n$.

$\therefore$ there exists at least one square submatrix R or order r such that $|R| \neq 0$.

Now R' is also a square submatrix of A' of order r.

Also $|R'| = |R| \neq 0$

$\therefore \quad \rho(A') \geq r$

If possible, suppose $\rho(A') > r$

We take $\rho(A') = r+1 \Rightarrow \rho(A') \geq r+1$ [$\because$ (A')' = A]

$\Rightarrow \rho(A) \geq r+1$ [$\because$ (A')' = A]

which is impossible as $\rho(A) = r$

$\therefore$ our supposition is wrong

$\therefore \rho(A') \not> r$

from (1), we have.

$\rho(A') = r \Rightarrow \rho(A') = \rho(A)$.

Result 2 : Prove that $d\,\rho(\lambda A) = \rho(A)$ where λ is a non-zero scalar.

Proof : If A = O, then $\lambda A = O$

$\therefore$ $\rho(A) = 0$ and $\rho(\lambda A) = 0$

$\therefore$ $\rho(\lambda A) = \rho(A)$

$\therefore$ result is true in the case in which A is a zero matrix.

Now we discuss the case when $A \neq O$.

Let r be the rank of the matrix $A = [a_{ij}]$ where A is of type m × n.

$\therefore$ there exists at least one square submatrix R of order r such that $|R| \neq 0$

Now λ R is a square submatrix of matrix λ A of order r.

$\therefore \quad |\lambda R| = \lambda^r |R| \neq 0$ as $\lambda \neq 0, |R| \neq 0$

$\therefore \rho(\lambda A) \geq r$ then $\rho(\lambda A) \geq r$ ...(1)

If possible, suppose $\rho(\lambda A) > r$

We take $\rho(\lambda A) > r+1$

$\therefore \rho\left(\frac{1}{\lambda}(\lambda A)\right) \geq r+1$ $[\because$ of (1)]

$\Rightarrow$ $\rho(A) \geq r+1$, which is impossible as $\rho(A) = r$

$\therefore$ Our supposition is wrong

$\because$ $\rho(\lambda A) = r$ or $\rho(\lambda A) = \rho(A)$

Result 3 : If A is an n-rowed non-singular matrix, then prove that $d\rho\left(A^{-1}\right) = \rho(A)$.

Hence deduce that $\rho(\text{adj.}A) = \rho(A)$.

Proof : Here A is an n-rowed non-singular matrix

$\therefore$ $|A| \neq 0$

$\Rightarrow$ $\rho|A| = n$

$\because$ A is non-singular

$\therefore$ A^{-1} exists and $AA^{-1} = 1$

$\Rightarrow \left|AA^{-1}\right| = |I| \Rightarrow |A|\left|A^{-1}\right| = 1 \Rightarrow \left|A^{-1}\right| \neq 0$

$\therefore$ A^{-1} is an n-rowed non-singular matrix

$\Rightarrow$ $\rho\left(A^{-1}\right) = n \Rightarrow \rho\left(A^{-1}\right) = \rho(A)$

Deduction

$$\rho(\text{adj.}A) = \rho\left(\frac{1}{|A|}\text{adj.}A\right) \qquad [\because \rho(A) = \rho(\lambda A)]$$

$$= \rho\left(A^{-1}\right) = \rho(A) \Rightarrow \rho(\text{adj.}A) = \rho(A).$$

Problem 1 : Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

Since there does not exist any minor of order 4 or A

$\therefore \quad \rho(A) \le 3 \qquad \ldots(1)$

$$\text{Now} \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 11 \end{vmatrix} = 1\begin{vmatrix} 3 & -4 \\ 3 & 11 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -4 \\ 5 & 11 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix}$$

$\therefore$ there exists a minor of order 3 of A which does not vanish.

$\therefore \rho(A) \ge 3$

From (1) and (2), we get,

$\rho(A) = 3$.

2.1.4 Elementary Operations (or Transformations)

We can also determive the rank of a matrix by using some other methods which are bared on the element ary transformations of a matrix, that includes :

(1) The interchange of any two parallel lines.

(2) The multiplication of all the elements of any line by any non-zero number.

(3) The addition to the elements of any line, the corresponding elements of any other line multiplied by any number.

Note : An elementary transformation is called a row transformation or a column transformation according as it applies to rows or columns. Therefore, there are three row transformation and three column transformations.

Symbols used for the transformations

(1) R_{ij} or $R_i \leftrightarrow R_j$ stands for the interchange of the ith and jth rows.

(2) $R_i^{(c)}$ or $R_i \to cR_i$ stands for the multiplication of the ith row by $c \ne 0$.

(3) $R_{ij}^{(k)}$ or $R_i \to R_i + kR_j$ stands for addition to the ith row, the product of the jth row by k. Similarly

(4) C_{ij} or $C_i \leftrightarrow C_j$ stands for the interchange of ith and jth columns.

(5) $C_i^{(c)}$ or $C_i \to cC$, stands for the multiplication of the elements of the ith column by $c \neq 0$.

(6) $C_{ij}^{(k)}$ or $C_i \to C_i + kC_j$ stands for addition to the ith column, the product of the jth column by k.

Definition of Elementary Matrix

A matrix, obtained from a unit matrix, by subjecting it to a single elementary transformation is called an elementary matrix.

Remarks :

2.1.5 Determination of Rank by Equivalent Matrix.

When an elementary transformation is applied to a matrix, it results into a matrix of the same order and same rank. The resulted matrix said to be equivalent to the given matrix and we use the symbol - to mean.

Let A be any given matrix, Reduce the matrix to equivalent matrix by using the following steps :

(i) Use row or column transformations, if necessary, to obtain a non-zero element (preferably 1) in the first row and the first column of the given matrix.

(ii) Divide the first row by this element, if it is not 1.

(iii) Subtract suitable multiples of the first row from the other rows so as to obtain zeros in the remainder of the first column.

(iv) Subtract suitable multiples of the first column from the other columns so as to get zeros in the remainder of the first row.

(v) Repeat the steps (i) – (iv) starting with the elements in the second-row and the second column.

(vi) Continue in this way down the "main diagonal" either until the end of the diagonal is reached or until all the remaining elements in the matrix are zero. The rank of this matrix, which is equivalent to the given matrix A, can be determined by inspection and consequently the rank of the given matrix A can be determined.

Problem 2 : Using elementary transformations, find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -3 \\ 0 & -4 & -2 \end{bmatrix}, \text{ by } R_2 \to R_2 - 4R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \text{ by } R_2 \to -\frac{1}{3}R_2, R_3 \to -\frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } C_2 \to C_2 - 3C_1, C_3 \to C_3 - 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } C_3 \to C_3 - \frac{1}{2}C_2$$

The rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2 as minor $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$ of order 2 does not vanish

$\therefore \rho(A) = 2$.

Problem 3 : Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$$

Sol. $A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{by } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } C_2 \to C_2 + C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - 5C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 - C_2, C_4 \to C_4 + 4C_2$$

The rank of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$ is 3 as the minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -6 \neq 0$ of order 3 does not vanish

$\therefore \quad \rho(A) = 3.$

Note : Normal form of a Matrix : The normal form of matrix A can be

$\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}, [I_r \quad O], \begin{bmatrix} I_r \\ O \end{bmatrix}$, where I_r is identity matrix of order 'r'.

Remarks : 1. Every non-zero matrix of rank r can, by a sequence of elementary transformations, be reduced to the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ where I, is a r-rowed unit matrix.

2. Let A be any non-zero matrix of rank r. Then there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

3. A non-singular matrix can be reduced to a unit matrix by a series of elementary transformations.

4. Every non-singular matrix is a product of elementary matrices.

5. The rank of a matrix is not altered by pre-multiplication or post-multiplication of the matrix with any non-singular matrix.

6. The rank of a product of two matrices cannot exceed the rank of either matrix.

Problem 4 : Prove that the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is equivalent to I_3.

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 1 & 2 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -6 \\ 0 & 1 & 2 \end{bmatrix}, \text{ by } C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix}, \text{ by } R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & -4 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \text{ by } C_3 \to C_3 - 6C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } R_3 \to -\frac{1}{4} R_3$$

$\therefore A \sim I_3$

$\therefore$ given matrix is equivalent to I_3.

Problem 5 : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, then find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of the matrix A.

Sol. Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$,

We have A = I A I

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } R_2 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

by $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } R_2 \to -\frac{1}{2}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } R_3 - R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ by } C_3 \to C_3 - C_2$$

$$\Rightarrow \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & --1 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ PAQ is in the normal form and $\rho(A) = 2$.

Problem 6 : Reduce the matrix $\begin{bmatrix} 2 & -1 & 0 & 4 \\ 1 & 3 & 5 & -3 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}$ to normal form. Hence find the rank of the matrix.

Solution : $A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 1 & 3 & 5 & -3 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -3 \\ 2 & -1 & 0 & 4 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}, \text{by } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -3 \\ 0 & -7 & 10 & 10 \\ 0 & -14 & -20 & 20 \\ 0 & -14 & -20 & 20 \end{bmatrix}, \text{by } R_1 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_3 \to R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 10 & 10 \\ 0 & -14 & -20 & 20 \\ 0 & -14 & -20 & 20 \end{bmatrix}, \text{by } C_2 \to C_2 - 3R_1, C_3 \to C_3 - 5C_1, C_4 \to C_4 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}, \text{by } C_2 \to -\frac{1}{7}C_2, C_3 \to -\frac{1}{10}C_3, C_4 \to \frac{1}{10}C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 \to 2R_2, R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 - C_2, C_4 \to C_4 - C_2$$

$$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

$\therefore$ rank (A) = 2

Self Check Exercise

1. Find the rank of each matrix $\begin{bmatrix} 0 & 6 & 6 & 1 \\ -8 & 7 & 2 & 3 \\ -2 & 3 & 0 & 1 \\ -3 & 2 & 1 & 1 \end{bmatrix}$.

....

....

....

2.1.6 Computation of Inverse using Elementary Transformations

We can understand this compoutation of finding invese with the help of following example:

If we are to find inverse of A, we write A = I A and go on performing row transformations on the product and the prefactor of A till we reach the result I = BA, then B is the inverse of A.

Problem 7 : Using elementary operations, find inverse of the matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$

Now $A = I\,A$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ -3 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A, \text{by } R_2 \leftrightarrow R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & -11 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} A, \text{by } R_2 \to R_2 + 3R_1, R_3 \to R_3 - 2R_1$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & -286 & -78 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 26 & 0 & -52 \end{bmatrix} A, \text{by } R_3 \to 26R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 26 & 11 & -19 \end{bmatrix} A, \text{by } R_3 \to R_3 + 11R_2$$

$$\therefore \begin{bmatrix} 1 & 7 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 52 & 22 & -37 \\ 182 & 78 & -130 \\ 26 & 11 & -19 \end{bmatrix} A, \text{by } R_1 \to R_1 + 2R_3, R_2 \to R_2 + 7R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 52 & 22 & -37 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix} A, \text{by } R_2 \to \frac{1}{26} R_2, R_3 \to -R_3$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix} A, \text{by } R_1 \to R_1 - 7R_2$$

$$\therefore \quad I = A^{-1} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix}.$$

Self Check Exercise

1. Use elementary transformation to find the inverse of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

....
....
....

2.1.7 Summary

In this lesson, we have gained knowledge about the rank of a matrix and learnt its evaluation using elementary transformations. We have also explained about the calculation of inverse of a matrix using elementary transformations. The concept is made more elaborative with the help of various suitable examples.

2.1.8 Key Concepts

Rank of a matrix, Equivalent matrices, Normal form.

2.1.9 Long Questions

1. Show that the matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if

$a^2 + b^2 + c^2 + d^2 = 1 \cdot$

2. (i) If P, Q are unitary, prove that QP is also unitary.

(ii) If P, Q are orthogonal, prove that QP is also orthogonal.

3. If A is an orthogonal matrix, then A' and A^{-1} are also orthogonal.

4. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -4 & 1 & 2 \\ 5 & 2 & 1 & 3 \end{bmatrix}$.

5. Find the rank of the matrix $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 7 & 6 & 2 & 5 \end{bmatrix}$, using equivalent matrix.

6. For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ is in the normal form and hence find out rank of matrix A.

7. Reduce the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$ to the form I_3 and find rank.

2.1.10 Short Questions

1. Find the rank of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

2. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, using equivalent matrix.

3. Use elementary transformation to find the inverse of

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 5 & 2 & 3 \end{bmatrix}$$

2.1.11 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

LESSON NO. 2.2 **AUTHOR : DR. CHANCHAL**

ROW RANK, COLUMN RANK AND THEIR EQUIVALENCE

2.2.1 Objectives

For this lesson, our prime objectives are

- To discuss about row rank and column rank of a matrix and their equivalence
- To discuss methods for checking linear dependence/independence of vectors

2.2.2 Row and Column Rank of a Matrix.

Firstly, we define the echelon form of a martrix:

A matrix A is said to be a row (column) equivalent to a matrix B if B can be obtained from A after a finite number of elementary row (column) operations, and we write $A \overset{R}{\sim} B$ or $A \overset{C}{\sim} B$.

Definition (Echelon Form):

A matrix A = $[a_{ij}]$ is said to be in the echelon form it

(i) The zero rows (columns) of A occur below all the non-zero rows (columns) of A

(ii)The number of zeros before the first non-zero element in a row (column) is less than the number of such zeros in the next row (column).

(iii) If $R_1, R_2, ...$ are non-zero rows (columns) of A, then first non-zero entry in these rows (columns) is 1. The Moreover Matrix is in (column) row reduced echelon form in addition to the above conditions, if a column (row) contains the first non-zero entry of any row(column), then every other entry in that column (row) is zero.

Row and column Rank of a Matrix

Let A be any matrix. Then Row rank of A, denoted by $\rho_R(A)$, is defined as the number of non-zero rows in a row echelon form of A.

Similarly,m column rank of A, denoted by $\rho_C(A)$, is defined as the number of non-zero column in a column echelon form of A.

Problem 1 : Find the row rank and column rank of $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -13 & -10 & -19 \\ 0 & -13 & -10 & -19 \\ -0 & 26 & 20 & 38 \end{bmatrix}, \text{by } R_2 \to R_2 - 5R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 + 7R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -13 & -10 & -19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 - R_2, R_4 \to R_4 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & \frac{10}{13} & \frac{19}{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_2 \to -\frac{1}{13}R_2$$

Which is in row-echelon form.

Since there are two non-zero rows in the row-echelon form

$\therefore$ row rank of A is 2

Now $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & -13 & -10 & -19 \\ 3 & -13 & -10 & -19 \\ -7 & 26 & 20 & 38 \end{bmatrix}, \text{by } C_2 \to C_2 - 3C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 - 4C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & -10 & -19 \\ 3 & 1 & -10 & -19 \\ -7 & -2 & 20 & 38 \end{bmatrix}, \text{by } C_2 \to C_2 - \frac{1}{13}C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -7 & -2 & 0 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 + 10C_2, C_4 \to C_4 + 15C_2$$

which is column echelon form having two non-zero column.

$\therefore$ column rank = 2.

In order to understand the concept of row rank and column rank more deeply, we must have the knowledge of law vectors and column vectors.

2.2.3 Linear Dependence | Independence of Vectors

Definition (n-vector) : An ordered tuple of n numbers is called n-vector.

For example : $\{x_1, x_2, \ldots.., x_n\}$ is an n-vector.

Linear Dependence of Vectors : A set V_1, V_2, …, V_t of vectors is said to be linearly dependent set, if there exists t scalars p_1, p_2, …, p_i, not all zero, such that

$p_1V_1 + p_2V_2 + \ldots + p_tV_t = O,$ where O is a n-vector with all components zero.

Any set of vectors, which is not linearly dependent, is called linearly independent.

i.e. a set $V_1, V_2, \ldots, V_t$ of n-vectors is said to be linearly independent if every relation of the form $p_1V_1 + p_2V_2 + \ldots + p_tV_t = O$ implies $p_1 = p_2 = \ldots = p_t = 0$.

Linear Combination of Vectors :

A vector V is said to be a linear combination of the vectors

$V_1, V_2, \ldots, V_t$ if $V = p_1V_1 + p_2V_2 + \ldots + p_tV_t,$ where $p_1, p_2, \ldots, p_t$ are scalars.

Result 1 : If a set of vectors is linearly dependent, show that at least one member of the set is a linear combination of the remaining members.

Proof : Let $V_1, V_2, \ldots, V_t$ be any linearly dependent set.

$\therefore$ the relation $p_tV_t + p_2V_2 + \ldots + p_tV_t = O$

implies that at least one of $p_1, p_2, \ldots, p_t$ is non-zero

Let p_t be non-zero

Now $p_1V_1 = -p_2V_2 - p_3V_3 - \ldots - p_tV_t$

$$\therefore V_t = \left(-\frac{p_2}{p_1}\right)V_2 + \left(-\frac{p_3}{p_1}\right)V_3 + \ldots + \left(-\frac{p_t}{p_1}\right)V_t$$

The relation (1) shows that V_1 is a linear combination of $V_2, V_3, \ldots, V_t$.

Result 2 : If η is a linear combination of the set $\{V_1, V_2, \ldots, V_r\}$, then the set $\{\eta, V_1, V_2, \ldots, V_r\}$ is linearly dependent.

Proof : Since η is a linear combination of $V_1, V_2, \ldots, V_r$

$\therefore\ \eta = k_1V_1 + k_2V_2 + \ldots + k_rV_r$

$\Rightarrow \eta - k_1V_1 - k_2V_2 - \ldots - k_rV_r = O$

Now at least one of the coefficients i.e. of η is non-zero.

$\therefore$ set $\eta, V_1, V_2, \ldots, V_r$ is L.D.

Result 3 : Prove that every super set of a linearly dependent set is linearly dependent.

Proof : Let $\{V_1, V_2, \ldots, V_p, V_{p+1}, \ldots, V_r\}$ be a super set of a linearly dependent set $\{V_1, V_2, \ldots, V_p\}$. Since $\{V_1, V_2, \ldots, V_p\}$ is linearly dependent set

$\therefore$ there exist scalars $k_1, k_2, \ldots, k_p$ (not all zero) such that

$k_1V_1 + k_2V_2 + ... + k_pV_p = O$

It can be re-written as

$k_1V_1 + k_2V_2 + ... + k_pV_p + ... + k_rV_r = O,$ where $k_1, k_2, ..., k_p, ..., k_r$ and not all zero.

$\therefore$ set $\{V_1, V_2, ..., V_p, ..., V_r\}$ is L.D.

Brief Outline of Vector Space :

In this lesson we briefly explain what a vector space is? The detailed studyof vector space will be done in lesson 0.5.

Definition : The n-vector Space : The set of all n-vectors over a field F, to be denoted by V_n(F), is called the n-vector space over F.

Sub-space of n-vector Space V_n

Any non-zero empty set, S of vectors of V_n(F) is called a subspace of V_n(F), if when

(i) V_1, V_2 are any two members of S, then $V_1 + V_2$ is also a member of S and

(ii) If V is a member of S and k is a member of F, then kV is also a member of S.

Subspace Spanned by a Set of Vectors

Let V_1, V_2, ..., V_t be a set of n-vectors.

The set of all linear combinations of the above set is called a subspace spanned by the set of vectors V_1, V_2, ..., V_t.

Basis of a Subspace

A set of vectors is said to be the basis of a subspace, if

(i) the subspace is spanned by the set and

(ii) the set is linearly independent.

Dimension of a subspace

The number of vectors in any basis of a subspace is called the dimension of the subspace.

Another Method to check for the Linear Dependence of Vectors :

Let $V_1 = (b_{11}, b_{12}, ..., b_{1n}), V_2 = (b_{21}, b_{22}, ..., b_{2n})$

....

....

$v_n = (b_{n1}, b_{n2}, ..., b_{nn})$ be a vectors of the vector space.

By definition these vectors are L.D. vectors iff there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero such that $\alpha_1V_1 + \alpha_2V_2 + \alpha_nV_n = O$

$$\Rightarrow a_1(b_{11}, b_{12}, ..., b_{1n}) + \alpha_2(b_{21}, b_{22}, ..., b_{2n}) + ... + \alpha_n(b_{n1}, ..., b_{nn}) = O$$

$$\Rightarrow (\alpha_1 b_{11} + \alpha_2 b_{21} + \ldots + \alpha_n b_{n1}, \alpha_1 b_{12} + \alpha_2 b_{22} + \ldots..$$

$$+\alpha_n b_{n2}, \ldots, \alpha_1 b_{1n} + \alpha_2 b_2 n + \ldots + \alpha_n b_{nn})$$

$$= (0, 0, \ldots, 0)$$

$$\therefore \quad \alpha_1 b_{11} + \alpha_2 b_{21} + \ldots + \alpha_n b_{n1} = 0$$

$$\alpha_1 b_{12} + \alpha_2 b_{22} + .. + \alpha_n b_{n2} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha_1 b_{1n} + \alpha_2 b_{2n} + \ldots + \alpha_n b_{nn} = 0$$

These homogenous equations must have a non-trivial (α_i's not all zero) solution. Moreover the above equations will have a non-trivial solution iff the determinant of its coefficient matrix is zero

$$\text{i.e.,} \quad \text{iff} \begin{vmatrix} b_{11} & b_{21} & \ldots & b_{n1} \\ b_{12} & b_{22} & \ldots & b_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ b_{1n} & b_{2n} & \ldots & b_{nn} \end{vmatrix} = 0$$

Hence vectors $V_1, V_2, \ldots, V_n$ are L.D.

$$\text{iff} \quad \begin{vmatrix} b_{11} & b_{21} & \ldots & b_{n1} \\ b_{12} & b_{22} & \ldots & b_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ b_{1n} & b_{2n} & \ldots & b_{nn} \end{vmatrix} = 0$$

$$\text{or} \quad \text{iff} \begin{vmatrix} b_{11} & b_{21} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{vmatrix} = 0$$

[$\because$ value of det. remains unchanged if rows and columns are interchanged] and vectors are L.I. iff this determinant $\neq 0$.

Problem 2 : Examine whether (1, –3, 5) belongs to the linear space generated by S, where $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$ or not?

Sol. If possible, let (1, –3, 5) belong to the linear space generated by S

$\Rightarrow \exists$ scalars α_1, α_2 and α_3 such that

$$(1,-3,5) = \alpha_1(1,2,1) + \alpha_2(1,1,-1) + \alpha_3(4,5,-2)$$

$$= (\alpha_1, 2\alpha_1, \alpha_1) + (\alpha_2, \alpha_2 - \alpha_2) + (4\alpha_3, 5\alpha_3, -2\alpha_3)$$

$$= (\alpha_1 + \alpha_2 + 4\alpha_3, 2\alpha_1 + \alpha_2 + 5\alpha_3, \alpha_1 - \alpha_2 - 2\alpha_3)$$

By equality of vectors, we must have

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 1$$

$$2\alpha_1 + \alpha_2 + 5\alpha_3 = -3$$

$$\alpha_1 - \alpha_2 - 2\alpha_3 = 5$$

Adding (1) and (3), we get,

$$2\alpha_1 + 2\alpha_3 = 6$$

$$\Rightarrow \alpha_1 + \alpha_3 = 3$$

and adding (2) and (3), we get,

$$3\alpha_1 + 3\alpha_3 = 2$$

$$\Rightarrow \alpha_1 + \alpha_3 = \frac{2}{3}$$

From (4) and (5), it is clear that we cannot found α_1 and α_2 and so α_3.

$\therefore$ our supposition is wrong.

Hence (1, –3, 5) does not belong to the Linear Space of S.

Problem 3 : Is the system of vectors [–1, 1, 2], [2, –3, 1], [10, –1, 0] linearly dependent ?

Sol. Given vectors are

$$V_1 = [-1,1,2],\ V_2 = [2,-3,1],\ V_3 = [10,-1,0]$$

Consider the relation

$$k_1V_1 + k_2V_2 + k_3V_3 = O$$

or $\quad k_1[-1,1,2] + k_2[2,-3,1] + k_3[10,-1,0] = [0,0,0]$

$$\therefore \quad -k_1 + 2k_2 + 10k_3 = 0 \qquad \dots (1)$$

$$k_1 - 3k_2 - k_3 = 0 \qquad \dots (2)$$

$$2k_1 + k_2 = 0 \qquad \dots (3)$$

From (3), $k_2 = -2k_1$

$\therefore$ from (2), $k_1 + 6k_1 - k_3 = 0 \Rightarrow k_3 = 7k_1$

$\therefore$ from (1), $-k_1 - 4k_1 + 70k_1 = 0 \Rightarrow 65k_1 = 0 \Rightarrow k_1 = 0$

$\therefore$ $k_2 = 0, k_3 = 0$

$\therefore$ $k_1 = k_2 = k_3 = 0$

$\therefore$ $k_1V_1 + k_2V_2 + k_3V_3 = O \Rightarrow k_1 = k_2 = k_3 = 0$

$\therefore$ given set of vectors is L.I.

Problem 4 : Find the value of k so that the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{and} \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} \text{are L.D.}$$

Sol. Let a, b, c be scalars, not all zero, such that

$$a\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} = O$$

where O is 3 × 1 zero matrix

$$\Rightarrow \begin{bmatrix} a \\ -a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ -2b \end{bmatrix} + \begin{bmatrix} ck \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + b + ck \\ -a + 2b \\ 3a - 2b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $a + b + ck = 0$

$-a + 2b = 0$

$3a - 2b + c = 0$

From (2), we get a = 2b

$\therefore 3 \Rightarrow 3a - a + c = 0 \Rightarrow c = -2a$

Put the values of c and b in (1), we have

$$a+\frac{a}{2}-2ak=0 \Rightarrow \frac{3a}{2}-2ak=0 \Rightarrow 2a\left(\frac{3}{4}-k\right)=0$$

But $a \neq 0$

[as if a = 0 then b = 0 and c = 0 which implies the given vectors are L.I.)

$$\Rightarrow \frac{3}{4}-k=0 \Rightarrow k=\frac{3}{4}.$$

2.2.4 Equality of Row Rank and column Rank Raw Rank and column Rank:

If A is any m × n matrix, then

(i) the space spanned by the set of m rows is called row space of A and the number of independent row vectors is called the row rank of A.

(ii) the space spanned by the set of n columns is called Column Space of A and the number of independent column vectors is called the column rank of A.

In other words, Column rank of any matrix A is the maximum number of linearly independent columns of A.

Result 4 : Prove that pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.

Proof : Let $A=\begin{bmatrix} R_1 \\ R_2 \\ : \\ R_m \end{bmatrix}$ be m × n matrix and

$$P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & P_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$ be m × n non-singular matrix

Let $$B=PA=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & P_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \\ : \\ R_m \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}R_1 + p_{12}R_2 + ... + p_{1m}R_m \\ p_{21}R_1 + p_{22}R_2 + ... + p_{2m}R_m \\ .... \\ p_{m1}R_1 + p_{m2}R_2 + ... + p_{mm}R_m \end{bmatrix}$$

$\therefore$ each row of B is a linear combination of the rows R_1, R_2, ...R_m of matrix A.

$\therefore$ every member of the row space of B is as will a member of the row space of the matrix A.

Again $\qquad A = P^{-1} B$

Proceeding as above, we find that every member of the row space of A is a member of the row space of B.

$\therefore$ row spaces of A and B are the same

$\therefore$ pre-multiplication with a non-singular matrix does not alter the row rank.

Note : We can prove that post-multiplication with a non-singular matrix does not alter the column rank of a matrix.

Result 5 : If s, be the row rank of an m × n matrix A, then there exists a non-singular matrix R of type m × m, such that

$$RA = \begin{bmatrix} K \\ O \end{bmatrix}$$

where K is an s × n matrix consisting of a set of a linearly independent rows of A.

Proof : Here A an s × n matrix with s as row rank, so out of m rows of A, s rows are L.I and remaining m-s are L.D. By row transformations, being s L.I. rows in first s rows. Now each of the last (m – s) rows, which are L.D., is a linear combination of first s rows.

Now subtracting suitable multiples of first s rows from the last (m – s) rows, we get a matrix in which each of last (m – s) rows is a zero row. Therefore the resulting matrix is of the type $\begin{bmatrix} K \\ O \end{bmatrix}$.

We know that every row transformation can be effected by pre-multiplying with a non-singular matrix. Let R be the product of all non-singular matrices corresponding to all row transformations.

$$\therefore \; RA = \begin{bmatrix} K \\ O \end{bmatrix}, \text{ where K is s × n matrix.}$$

Note : Similarly we can prove that $AS = [L \quad O]$, where L is an m × s matrix, s being the column rank of A.

Result 6 : Prove that the row rank of a matrix is the same as its rank.

Proof : Let r be the rank and s be the row rank of m × n matrix A.

Since s is row rank of A

∴ there exists a non-singular matrix R such that

$RA = \begin{bmatrix} K \\ O \end{bmatrix}$, where K is s × n matrix

since each minor of order (s + 1) of the matrix RA involves at least are row of zeros

∴ $\rho(RA) \leq s$

∴ $r \leq s$ ...(1)

Since r is rank of A

∴ there exists a non-singular matrix P such that

$PA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is r × n matrix.

The row rank of PA, being the same as that A, is s. Also PA has only r non-zero rows.

∴ the row rank of PA can, at the most be r

∴ $s \leq r$ ..(2)

From (1) and (2)

r = s

i.e. rank of A = row rank of A.

Corellary : Prove that the column rank of a matrix is the same as its rank.

Proof : We know that columns of A are the rows of A'.

∴ column rank of A = row rank of A'

= rank of A'

= rank of A

Hence the result

Remarks :

1. Rank of A = row rank of A = column rank of A.

2. The rank of a matrix is equal to the maximum number of its linearly independent rows and also to the maximum number of its linearly independent columns.

3. If A an n-rowed non-singular matrix, then its rows as well as columns form L.I. sets.

4. If A, B be two matrices of the same type, then $\rho(A+B) \le \rho(A)+\rho(B)$.

5. If A, B are two n-rowed square matrices, then

$$\rho(AB) \ge \rho(A)+\rho(B)-n.$$

Problem 5 : Examine the linear independent or dependence of the rows of the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$, hence find its rank.

Solution : $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

$$\therefore |A| = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix} = 3\begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$= 3(0 + 2) - 2(-1 -2) + 4(-1-0)$

$= 3(2) - 2(-3) + 4(-1) = 6 + 6 - 4 = 8 \neq 0$

$\therefore$ A is non-singular matrix.

$\therefore$ three rows of A and L.I.

$\therefore$ rows of matrix A form of L.I. set

$\therefore$ $\rho(A) = 3$.

Problem 6 : Find the value of k so that the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{and} \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} \text{are L.D.}$$

Solution : Let a, b, c be scalars, not all zero, such that

$$a\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} = O$$

where O is 3 × 1 zero matrix

$$\Rightarrow \begin{bmatrix} a \\ a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ -2b \end{bmatrix} + \begin{bmatrix} ck \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b+ck \\ -a+2b \\ 3a-2b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $a + b + ck = 0$

$-a + 2b = 0$

$3a - 2b + c = 0$

From (2), we get $a = 2b$

$\therefore$ (3) $\Rightarrow 3a - a + c = 0 \quad \Rightarrow c = -2a$

Put the values of c and b in (1) , we have

$$a + \frac{a}{2} - 2ak = 0 \Rightarrow \frac{3a}{2} - 2ak = 0 \Rightarrow 2a\left(\frac{3}{4} - k\right) = 0$$

But $a \neq 0$

[as if a = 0 then b = 0 and c = 0 which implies the given vectors are L.I.]

$$\Rightarrow \frac{3}{4} - k = 0 \Rightarrow k = \frac{3}{4}.$$

Self Check Exercise

1. Show that the row vectors of the matrix $\begin{bmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{bmatrix}$ are linearly independent.

....

....

....

2.2.5 Summary

This lessons helps us to understand about the concept of linear combination of vectors and the vectors generating a subspace i.e. the vectors which form basis of a subspace. For this, we discussed about the linear dependence\independence of vectors. The concept is made more elaborative with the help of various suitable examples.

2.2.6 Key Concepts

Row rank, Column rank, Linear combination, Linear dependence/independence.

2.2.7 Long Questions

1. Reduce to row reduced echelon form the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 & 4 \\ 2 & 0 & -4 & 1 & 2 \\ 1 & 4 & 2 & 0 & -1 \\ 3 & 4 & -2 & 1 & -1 \\ 6 & 9 & -1 & 1 & 6 \end{bmatrix} \text{ and find } \rho_R(A).$$

2. Find the row rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

3. Show that the vectors $V_1 = (1,2,3), V_2 (0,1,2)$ and $V_3 = (0,0,1)$ generate $V_3(R)$.

4. Examine for linear dependence the vectors $[1,\ 2,\ 4,], [2,\ -1,\ 3], [0,\ 1,\ 2], [-3,\ 7,\ 2]$ and find the relation if it exists.

5. Prove that the vectors $x = (1,0,0), y = (0,1,0); z = (0,0,1)$ and $w = (1,1,1)$ form a linearly dependent set, but any three of them are linearly independent.

6. Determine whether the following matrices have same column space or not ?

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 15 \end{bmatrix}.$$

2.2.8 Short Questions

1. Define row rank of a matrix.

2. Define column rank of a matrix.

3. Define linear combination of vectors. 4. Define basis of a subspace.

4. Show that the vectors $[1\ \ 2\ \ 3], [3\ \ -2\ \ -1], [1\ \ -6\ \ -5]$ form a L.I. system.

2.2.9 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

LESSON NO. 2.3 | **AUTHOR : DR. CHANCHAL**

EIGEN VALUES AND EIGEN VECTORS

2.3.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Eigen values and Eigen vectors of a matrix
- Diagonalizable matrix and its corresponding diagonal matrix
- Cayley-Hamilton theorem
- The concept of minimal polynomial and minimal equation

2.3.2 Introduction

An expression of the form d $A_0x^m + A_1x^{m-1} + A_2x^{m-2} + ... + A_{m-1}x + A_m$. where $A_0, A_1, A_2, ..., A_m$ are all square matrices of the some order n and m is a positive integer, is called a n-rowed matrix polynomial of degree m.

Note : Two matrix polynomials are said to be equal iff the coefficients of the like powers of x are the same.

1. Eigen values are also known as proper values, characteristic values, latent roots or spectral values. Similarly eigen vectors are also called proper vectors, characteristic vectors, latest vectors or spectral vectors.

2. The set of characteristic roots of a matrix A is called the spectrum of the matrix A.

2.3.3 Some Important Results

Result 1 : Prove that λ is an eigen value of n-rowed square matrix A over a field F it and only if $|A - \lambda I| = 0$.

Proof : (i) Assume that λ is an eigen value of A over F.

$\therefore$ there exists a non-zero column matrix X of type n × 1 such tat

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = O$$

$$\Rightarrow AX - \lambda IX = O$$

$$\Rightarrow (A - \lambda I)X = O$$

$$\Rightarrow |A - \lambda I| = 0 \qquad [\because X \neq O]$$

[$\because AX = O$ has a non-trivial solution iff |A| = 0]

(ii) Assume that $|A - \lambda I| = 0$

$\therefore$ $|A - \lambda I|X = O$ has a non-trivial solution

$\therefore$ $AX - \lambda IX = O$

or $AX - \lambda X = O$

or $AX = \lambda X$

Where X is a non-zero matrix

$\therefore$ λ is an eigen value of A over F.

Note. λ is an eigen value of A over F iff $A - \lambda I$ is a singular matrix.

Result 2 : If X is a characteristic vector of a matrix corresponding to the characteristic value λ, then kX is also a characteristic vector of A corresponding to the same charactristic value $\lambda(k \neq 0)$.

Proof : Since X is a characteristic vector of A corresponding to the characteristic value λ.

$\therefore$ $X \neq O$ and

$AX = \lambda X$

Now A(k X)= k(A X) = k(λ X) = λ(k X)

Now kX is a non-zero vector such that $A(kX) = \lambda(kX)$

∴ kX is a characteristic vector of A corresponding to the characteristic value λ.

Note : Corresponding to a characteristic value λ , there may corresponds more than one characteristic vectors.

Result 3 : If X be an eigen vector of the n-rowed square matrix A over a field F, thn X cannot correspond to two distinct eigen values.

Proof : Since X is an eigen vector of A over F.

∴ X is a non-zero column matrix of order n × 1.

Suppose eigen vector X corresponds to two eigen values λ_1, λ_2 of A.

$$\therefore \quad AX = \lambda_1 X \text{ and } AX = \lambda_2 X$$

$$\Rightarrow \lambda_1 X = \lambda_2 X$$

$$\Rightarrow (\lambda_1 - \lambda_2) X = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0$$

Hence the result.

Result 4 : Prove that any system of eigen vectors X_1, X_2, ...X_m corresponding respectively to a system of distinct eigen values $\lambda_1, \lambda_2, \lambda_m$ of a matrix A is linearly independent.

Proof : Try Yourself.

Result 5 : Prove that the characteristic roots of a hermitian matrix are real.

Proof : Let λ be a characteristic roots of a hermitian matrix A.

∴ there exists a non-zero n × 1 column matrix X such that

AX = λX

$$\Rightarrow X^\theta (AX) = X^\theta (\lambda X)$$

$$\Rightarrow X^\theta AX = \lambda X^\theta X$$

$$\Rightarrow \left(X^\theta AX\right) = \left(\lambda X^\theta X\right)^\theta$$

$$\Rightarrow X^\theta A^\theta \left(X^\theta\right)^\theta = \bar{\lambda} X^\theta \left(X^\theta\right)^\theta$$

$\Rightarrow X^\theta AX = \bar{\lambda}\, X^\theta X$ [$\because A^\theta = A$ as A is hermitian and $\left(X^\theta\right)^\theta = X$]

$\Rightarrow X^\theta \lambda\, X = \bar{\lambda}\, X^\theta X$ $[\because AX = \lambda X]$

$\Rightarrow \lambda X^\theta X = \bar{\lambda}\, X^\theta X$

$\Rightarrow \left(\lambda - \bar{\lambda}\right) X^\theta X = O$

$\Rightarrow \lambda - \bar{\lambda} = 0$ $\left[\because X^\theta \neq O \text{ as } X \neq O\right]$

$\Rightarrow \bar{\lambda} = \lambda$

$\Rightarrow \lambda$ is real

Hence the result.

Result 6 : Prove that any two characteristic vectors corresponding to two distinct characteristics roots of a hermitian matrix are orthogonal.

Proof : Let X_1, X_2 be the characteristic vectors corresponding to characteristic roots λ_1, λ_2 of the hermitian matrix A.

$\therefore$ $AX_1 = \lambda_1 x_1$

$AX_2 = \lambda_2 X_2$

From (1), $X_2{}^\theta AX_1 = X_2{}^\theta \lambda_1 X_1$

From (2), $X_1{}^\theta AX_2 = X_1{}^\theta \lambda_2 X_2$

Now $\left(X_2{}^\theta AX_1\right)^\theta = X_1{}^\theta A^\theta \left(X_2{}^\theta\right)^\theta = X_1{}^\theta AX_2$,

since $A^\theta = A$ as A is hermitian

$\therefore$ $\left(X_2{}^\theta \lambda_1 X_1\right)^\theta = X_1{}^\theta \lambda_2 X_2$

$\Rightarrow$ $\lambda_1 X_1{}^\theta \left(X_2{}^\theta\right)^\theta = \lambda_2 X_1{}^\theta X_2$ $[\because \lambda_1, \lambda_2 \text{ are real}]$

$\Rightarrow$ $\lambda_1 X_1{}^\theta X_2 = \lambda_2 X_1{}^\theta X_2$

$\Rightarrow$ $\left(\lambda_1 - \lambda_2\right) X_1{}^\theta X_2 = O$

But $\lambda_1 - \lambda_2 \neq 0$

$\therefore$ $X_1^{\theta} X_2 = 0$

$\Rightarrow$ X_1, X_2 are orhogonal.

Result 7 : Prove that characteristic roots of a unitary matrix are of unit modulus.

Proof : Let A be given unitary matrix.

$$\therefore \quad A^{\theta} A = 1 \qquad \ldots (1)$$

Let λ be a characteristic root of A.

$\therefore$ there exists a non-zero vector X such that

$$AX = \lambda X \qquad \ldots (2)$$

$$\Rightarrow \quad (AX)^{\theta} = (\lambda X)^{\theta}$$

$$\Rightarrow \quad X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta} \qquad \ldots (3)$$

From (2) and (3), we get,

$$\left(X^{\theta} A^{\theta}\right)(AX) = \left(\bar{\lambda} X^{\theta}\right)(\lambda X)$$

$$\Rightarrow \quad X^{\theta}\left(A^{\theta} A\right) X = \lambda\bar{\lambda}\, X^{\theta} X$$

$$\Rightarrow \quad X^{\theta} I X = \lambda\bar{\lambda}\, X^{\theta} X \qquad [\because \text{ of } (1)]$$

$$\Rightarrow \quad X^{\theta} X = \lambda\bar{\lambda}\, X^{\theta} X$$

$$\Rightarrow \quad \left(\lambda\,\bar{\lambda} - 1\right) X^{\theta} X = O$$

$$\Rightarrow \quad \lambda\bar{\lambda} - 1 = O \qquad \left[\because X^{\theta} X \neq O \text{ as } X \neq O\right]$$

$$\Rightarrow \quad |\lambda|^2 - 1 = O$$

$$\Rightarrow \quad |\lambda|^2 = 1$$

$$\Rightarrow \quad |\lambda| = 1$$

Hence the result.

Result 8 : Prove that any two characteristic vectors corresponding to two distinct characteristic roots of a unitary matrix are orthogonal.

Proof : Try Yourself.

2.3.4 Characterirtic Equation of a Matrix

If A be any n-rowed square matrix over a field F and λ an indeterminate, then the matrix $A-\lambda I$ is called the characteristic matrix of A.

The determinant $|A - \lambda I|$, an algebraic polynomial in λ of degree n, is called the characteristic polynomial of A.

The equation $|A - \lambda I| = 0$ is called characteristic equation of A.

Remark : An eigen value λ of matrix A is always a root of its characteristic equation and every root of the characteristic equation of A is an eigen value of A.

$\therefore$ in order to find eigen values of A, we should find roots of the characteristic equation of A.

2.3.5 Diagonalizable Matrix

An $n \times n$ matrix A is called diagonalizable if there exists an invertible $n \times n$ matrix P such that $P^{-1} AP$ is a diagonal matrix.

Method to find Diagonal Matrix for a Diagonalizable matrix.

Step I : Find eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ of A.

Step II : Find corresponding eigen vectors $X_1, X_2, \ldots, X_n$. If number of eigen vectors < n, A is not diagonalizable.

Step III : Find $P = \{X_1 X_2 X_3 \ldots X_n\}$ and P^{-1}.

Step IV : $p^{-1}AP = \text{Diag.}\left(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\right)$

is required diagonal matrix.

Note : A is diagonalizable if and only if A has n L.I. eigen vectors.

2.3.6 Cayley Havilton Theorem

Statement : Every square matrix satisfies its characteristic equation.

Proof : Let A be any square matrix of order n, and its characteristic equation be

$$p_0 + p_1\lambda + p_2\lambda^2 + \ldots + p_n\lambda^n = 0$$

We have to prove that A satisfies this equation

i.e., $p_1 1 + p_1 A + p_2\lambda^2 + \ldots + p_n A^n = O$

For proving this, we proceed as follows :

We know that $(A - \lambda I)\,\text{adj.}\,(A - \lambda I) = |A - \lambda|\, I \qquad [\because A\ \text{adj.}\ A = |A|\, I]$

Let adj. $(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1}$

$\therefore \qquad$ we have, $(A - \lambda I)\left(B_0 + B_1\lambda + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1}\right)$

$$= \left(p_0 + p_1\lambda + p_2\lambda^2 + \ldots + p_n\lambda^n\right)I$$

Equating the coefficients of like powers of λ, we get,

$$AB_0 = p_0 I$$

$$AB_1 - B_0 = p_1 I$$

$$AB_2 - B_1 = p_2 I$$

$$\ldots \ldots \ldots \ldots \ldots \ldots$$

$$AB_{n-1} - B_{n-2} = p_{n-1} I$$

$$-B_{n-1} = p_n I$$

Pre-multiplying above equations by I, A, A^2, ..., A^n respectively and adding, we get,

$O = p_0 I + p_1 A + p_2 A^2 + \ldots + p_n A^n$, which is same as (1).

Hence the theorem.

2.3.7 Minimal Polynomial and Minimal Equation

If m(x) be a scalar polynomial of the lowest degree with leading coefficient unity, such that m(x) = 0 is satisfied by A i.e. m(A) = O, then the polynomial m(x) is called the minimal polynomial of A and m(x) = 0 is called the minimum equation of A.

Note. The degree of the minimal equation of an n-rowed matrix is less than or equal to that of its characteristic equation which is n.

Derogatory and Non-derogatory Matrices

An n-rowed matrix is said to be derogatory or non-derogatory, according as the degree of its minimal equation is less than or equal to n.

2.3.8 Problems

Problem 1 : Prove that a square matrix A and its transpose A have the same set of eigen values.

Sol. Characteristic polynomial of A^t

$$= |A^t - \lambda I| = |A - \lambda I^t| = |(A - \lambda I)^t|$$

$$= |A - \lambda I| \qquad \left[\because |A^t| = |A|\right]$$

= characteristic polynomial of A

∴ A and A^t have same characteristic polynomial and hence the same set of eigen values.

Problem 2 : If α is an eigen value of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is an eigen value of adj. A.

Sol. Since α is an eigen of a non-singular matrix A

$\therefore$ $\alpha \neq 0$ and there exists a non-zero column vector X such that

$$AX = \alpha X$$

$$\Rightarrow \quad (\text{adj. } A)(AX) = (\text{adj. } A)(\alpha X)$$

$$\Rightarrow \quad [(\text{adj. } A)(A)]X = \alpha[(\text{adj. } A)X]$$

$$\Rightarrow \quad (|A|I)X = (\text{adj. } A)X$$

$$\Rightarrow \quad (\text{adj. } A)X = \frac{|A|}{\alpha}X$$

$\Rightarrow$ $\frac{|A|}{\alpha}$ is an eigen value of adj. A.

Problem 3 : The characteristic roots of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix}$ are 0, 3 and 15. Find the value of k.

Sol. Let $$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & k-\lambda \end{bmatrix}$$

$\therefore$ characteristics equation of matrix A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & k-\lambda \end{vmatrix} = 0$$

Since $\lambda = 0$ is a root of it

$$\therefore \quad \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{vmatrix} = 0$$

$$\therefore \quad 8\begin{vmatrix} 7 & -4 \\ -4 & k \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & k \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix} = 0$$

or $\quad 8(7k-16)+6(-6k+8)+2(24-14)=0$

or $\quad 56k-128-36k+48+20=0$

$\therefore \quad 20k=60 \Rightarrow k=3.$

Problem 4 : Define similar matrices and prove that similar matrices have same characteristic polynomial and hence same eigen values.

Sol. Let A and B be square matrices of order n over a field F. Then A is said to be similar to B over F if and only if there exists an n-rowed invertible matrix P over F such that

$$AP = PB \text{ i.e. } B = P^{-1}AP \text{ or } A = PBP^{-1}$$

Let A and B be two similar matrices

$\therefore \quad B = P^{-1}AP$

$\therefore \quad B-\lambda I = P^{-1}AP-\lambda I = P^{-1}AP-\lambda P^{-1}P \qquad [\because P^{-1}P = I]$

$\qquad = P^{-1}AP - P^{-1}(\lambda I)P = P^{-1}(A-\lambda I)P$

$\therefore \quad |B-\lambda I| = |P^{-1}(A-\lambda I)P|$

$\qquad = |P^{-1}||A-\lambda I||P|$

$\qquad = |A-\lambda I||P^{-1}||P| = |A-\lambda I||P^{-1}P|$

$\qquad = |A-\lambda I||I|$

$\therefore \quad |B-\lambda I| = |A-\lambda I| \qquad [\because |I| = 1]$

$\therefore$ matrices A and B = $P^{-1}AP$ have the same characteristic polynomial and hence the same set of eigen values.

Problem 5 : Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Is it diagonalisable ? Justify.

Sol. $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

$\therefore$ the characteristics equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0, \text{ by } R_1 \to R_1 + R_2 + R_3$$

or $$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \text{ or } (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

or $(6-\lambda)\left[(1)(2-\lambda)(2-\lambda)\right]=0$

or $(6-\lambda)(2-\lambda)^2=0$

$\therefore$ $\lambda = 2, 2, 6$

which are the eigen values of A.

The eigen vector $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq O$ corresponding to the eigen value $\lambda=6$ is given by

$$AX=\lambda X \text{ or } (A-6I)X=O$$

or $\begin{bmatrix}-3&1&1\\2&-2&2\\1&1&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}1&1&-3\\2&-2&2\\-3&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

or $\begin{bmatrix}1&1&-3\\0&-4&8\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, by $R_3\to R_3+R_2$

Now the coefficient matrix of these equations is of rank 2. Therefore these equations have only 3 – 2 = 1 L.I. solution. Thus there is only one L.I. eigen vector corresponding to the value 6. These equations can be written as

$x + y - 3z = 0$

$-4y+8z=0 \quad \Rightarrow y=2z$

$\therefore\ x+2z-3z=0 \quad \Rightarrow x=z$

Take $z = 1$, $\therefore x=1, y=2$

$X=\begin{bmatrix}1\\2\\1\end{bmatrix}$ is an eigen vector of A.

The eigen vector $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq O$ corresponding to the eigen value $\lambda=2$ is given by

$$AX = 2X \text{ or } (A = 2I)X = O$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

The coefficient matrix of these equations is of rank 1. Therefore these equations have 3 – 1 = 2 L.I. solutions. These equations can be written as

$x + y + z = 0$ or $x = -y - z$

Take $y = 1, z = 0$; $y = 0, z = 1$

Therefore we find two L.I. eigen vectors of A as $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\therefore \quad P = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|P| = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1-0) + 1(2-0) - 1(0-1)$$

$$= 1(1) + 1(2) - 1(-1) = 1 + 2 + 1$$

$$= 4$$

Co-factors of the elements of first row of $|P|$ are

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \text{ i.e. } 1, 2, 1 \text{ respectively}$$

Co-factors of the elements of second row of $|P|$ are

$$= \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \text{ i.e. } 1, 2, -1 \text{ respectively}$$

Co-factors of the elements of third row of |P| are

$$\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \text{ i.e. } 1, -2, 3 \text{ respectively}$$

$$\therefore \quad \text{adj}P = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 1 & -2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\therefore \quad p^{-1} = \frac{\text{adj}P}{|P|} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$p^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is a diagonal matrix.

Problem 6 : Verify Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$.

Hence find A^{-1} .

Solution : $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Rightarrow \lambda L=\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$\therefore \qquad A-\lambda I=\begin{bmatrix}-\lambda&0&1\\1&2-\lambda&0\\2&-1&-\lambda\end{bmatrix}=-\lambda\begin{vmatrix}2-\lambda&0\\-1&-\lambda\end{vmatrix}+1\begin{vmatrix}1&2-\lambda\\2&-1\end{vmatrix}$$

$$=-\lambda\left(-2\lambda+\lambda^2\right)+1\left(-1-4+2\lambda\right)=-\lambda^3+2\lambda^2-5+2\lambda$$

$$\therefore \qquad |A-\lambda I|=-\lambda^3+2\lambda^2+2\lambda-5$$

The characteristic equation of A is $|A-\lambda I|=0$

or $\quad -\lambda^3+2\lambda^2+2\lambda-5=0$ or $\quad \lambda^3-2\lambda^2-2\lambda-5=0$

We are to prove that A satisfies thiws equation i.e. $A^3-2A^2-2A+5I=O$

Now $A^2=\begin{bmatrix}0&0&1\\1&2&0\\2&-1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&2&0\\2&-1&0-\end{bmatrix}=\begin{bmatrix}2&-1&0\\2&4&1\\-1&-2&2\end{bmatrix}$

$$A^3=\begin{bmatrix}2&-1&0\\2&4&1\\-1&-2&2\end{bmatrix}\begin{bmatrix}0&0&1\\1&2&0\\2&-1&0-\end{bmatrix}=\begin{bmatrix}-1&-2&0\\6&7&2\\2&-6&-1\end{bmatrix}$$

Consider $A^3-2A^2-2A+5I$

$$=\begin{bmatrix}-1&-2&2\\6&7&2\\2&-6&-1\end{bmatrix}-2\begin{bmatrix}2&-1&0\\2&4&1\\-1&-2&2\end{bmatrix}-2\begin{bmatrix}0&0&1\\1&2&0\\2&-1&0\end{bmatrix}+5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\therefore \qquad A^3-2A^2-2A+5I=O$$

Re-multiplying both sides by A^{-1}, we get,

$$A^2-2A-2I+5I^{-1}=2I$$

$$= -\begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -1 \\ 5 & 0 & 0 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{5}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -1 \\ 5 & 0 & 0 \end{bmatrix}$$

Self Check Exercise

1. Verify Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Hence find A^{-1}.

....

....

....

2.3.9 Summary

In this lesson, we have studied about characteristic equation and the terms related to it. An important theorem based upon it i.e. Cayley-Hamilton theorem has been discussed. Moreover, the concept of diagonalizable matrix and to find out a diagonal matric for it, has been also elaborated. The concepts are made more clear with the help of various suitable examples.

2.3.10 Key Concepts

Characteristic roots, Characteristic equation, Diagonalizable matrix, Cayley-Hamilton theorem, Minimal polynomial, Minimal equation.

2.3.11 Long Questions

1. If λ is an eigen value of a square matrix A, then prove that $\bar{\lambda}$ is an eigen value of A^{θ} and conversely.

2. Show that the necessary and sufficient condition for a 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to have zero as an eigen value is that a d – b c = 0.

3. Diagonalize, if possibel, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$.

4. Using Cayley Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

2.3.12 Short Questions

1. Define diagonalizable matrix.

2. Discuss the concept of minimal polynomial and minimal equation.

3. Determine eigen values of the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$.

4. Find the characteristic roots and the spectrum of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

2.3.13 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.

2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

UPDATED ON APRIL, 2024

B.A. PART– I **MTHB1102T**

(SEMESTER - I) **ALGEBRA AND TRIGONOMETRY**

LESSON NO. 2.4 **AUTHOR : DR. CHANCHAL**

SYSTEM OF LINEAR EQUATIONS AND ITS CONSISTENCY

2.4.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Linearly independent solution for the system of homogeneous linear equations.
- Consistency of a system of non-homogeneous linear equations

2.4.2 Homogeneous and Non-Homogeneous Linear Equations (An Introduction)

Let
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_2 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right]$$

be a set of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$.

The above set of linear equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = O$$

i.e., $AX = O$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The matrix A is called the coefficient matrix.

Remarks : Any set of values $x_1, x_2, \ldots\ldots, x_n$ which satisfy simultaneously the m equtions in (1), is called a solution of the system.

A system of equations, which has a solution, is called consistent or compatible. If the system does not has any solution, it is called inconsistent.

2.4.3 Linearly Independent Solutions of AX = O

Conditions under which a set of homogeneous equations possess a (i) trivial solution of (ii) non-trivial solution.

Let there be m equations in n unknowns. So the coefficient matrix A is of type $m \times n$. Let r be rank of A.

Now either $r < n$ or $r = n$

(i) If $r = n$, then the equation $AX = O$ has $n - n = 0$

i.e. no linearly independent solution. Therefore, the equation $AX = O$ has trivial solution.

(ii) If $r < n$, then the equation $AX = 0$ has $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $AX = O$. So, there are infinite number of non-trivial solutions.

Article 1 : Let A be an $m \times n$ matrix of rank r. Then the equation $AX = O$ has $(n-r)$ linearly independent solutions.

Proof : The given equation is $AX = O$ ... (1)

We have to prove two results :

(i) $AX = O$ has $(n - r)$ solutions

(ii) $(n - r)$ solutions form a linearly independent set.

For proving first part, we proceed as following :

$\because$ rank of A = r

$\therefore$ column rank of A = r

$\therefore$ A has r linearly independent columns. We assume that first r columns are linearly independent and last (n–r) columns are linearly dependent.

Let $A = \begin{bmatrix} C_1 & C_2 \ldots C_r & C_{r+1} \ldots C_n \end{bmatrix}$

$\therefore$ equation (1) can be written as

$$\begin{bmatrix} C_1 & C_2 \ldots C_r & C_{r+1} \ldots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = O$$

i.e., $C_1x_1 + C_2x_2 + \ldots + C_rx_r + C_{r+1}x_{r+1} + \ldots + C_nx_n = O$

$\because$ $C_{r+1}, C_{r+1}, \ldots . C_n$ are linearly dependent columns.

$\therefore$ each is a linear combination of $C_1, C_2, \ldots \ldots, C_r$

Let $C_{r+1} = p_{11}\, C_1 + p_{12}\, C_2 + \ldots + p_{1r}\, C_r$

$C_{r+2} = p_{21}\, C_1 + p_{22}\, C_2 + \ldots + p_{2r}\, C_r$

....

$C_n = p_{t1}C_1 + p_{t2}\, C_2 + \ldots + p_{tr}C_r$ where t = n – r $\quad [\because n = r + (n-r)]$

The above equation can be written as

$$\left.\begin{array}{l} p_{11}C_1 + p_{12}C_2 + \ldots . + p_{1r}C_r + (-1)C_{r+1} + 0.C_{r+2} + \ldots . + 0.C_n = O \\ p_{21}C_1 + p_{22}C_2 + \ldots . + p_{2r}C_r + 0.C_{r+1} + (-1)C_{r+2} + \ldots . + 0.C_n = O \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{11}C_1 \; p_{12}C_2 \ldots . + p_{1r}C_r + 0.C_{r+1} + 0.C_{r+2} + \ldots . + (-1)\, C_n = O \end{array}\right]$$

Comparing one by one the equation in (3) with equation (2), we get,

$$X_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ : \\ p_{1r} \\ -1 \\ 0 \\ 0 \\ : \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} p_{21} \\ p_{22} \\ : \\ p_{2r} \\ 0 \\ -1 \\ 0 \\ : \\ 0 \end{bmatrix}, \ldots, X_t = \begin{bmatrix} p_{t1} \\ p_{t2} \\ : \\ p_{tr} \\ 0 \\ 0 \\ 0 \\ : \\ -1 \end{bmatrix}$$

as t = n – r solutions of the equation

Now we have to show that these n – r solutions $X_1, X_2, \ldots, X_t$ are linearly independent vectors.

For this we consider the relation

$$p_1X_1 + p_2X_2 + \ldots + p_tX_t = O$$

i.e.
$$p_1 \begin{bmatrix} p_{11} \\ p_{12} \\ : \\ p_{1r} \\ -1 \\ 0 \\ 0 \\ : \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} p_{21} \\ p_{22} \\ : \\ p_{2r} \\ 0 \\ -1 \\ 0 \\ : \\ 0 \end{bmatrix} + \ldots + p_t \begin{bmatrix} p_{t1} \\ p_{t2} \\ : \\ p_{tr} \\ 0 \\ 0 \\ 0 \\ : \\ -1 \end{bmatrix} = O$$

Comparing (r + 1)th, (r + 2)th ….. nth elements, we get,

$-p_1 = 0, -p_2 = 0, \ldots, -p_t = 0$

$\therefore \quad p_1 = p_2 = \ldots = p_t = 0$

$\therefore \quad p_1X_1 + p_2X_2 + \ldots.. + p_tX_t = 0$

$\Rightarrow \quad p_1 = p_2 = \ldots.. = p_t = 0$

$\therefore \quad X_1, X_2, \ldots\ldots, X_t$ are L.I. vectors

$\therefore$ A X = O has n – r L.I. solution.

Article 2 : The equation AX = O has a non-zero (i.e., non-trivial solution) iff A is singular.

Proof : Assume that AX = O has a non-zero solution.

$\therefore$ n – r > 0 where r is the rank of n-rowed matrix A implies n > r.

i.e., rank of A is less than the order of the matrix.

$\therefore$ A is a singular matrix.

Again, assume that A is a singular matrix

$\therefore$ |A| = 0

$\Rightarrow$ rank of A < order of A

$\Rightarrow$ r < n

$\Rightarrow$ n – r > 0

$\Rightarrow$ equation AX = 0 has a non-zero solution.

2.4.4 Consistency of AX = B

Conditions under which a system of non-homogeneous equations will have :

(i) no solution (ii) a unique solution (iii) infinity of solutions.

Let AX = B be a system of non-homogeneous equations.

(i) The equation AX = B has no soluiton if A and [A B] do not have the same rank.

(ii) The equation AX = B, has a solution if the rank of A is the same as that of [A B]. If in addition, A is non-singular, then equation has a unique solution.

(iii) The equation AX = B will have infinite solutions if A and [A B] have the same rank and A is singular.

Article 2 : The necessary and sufficient condition that the system of equations AX = B is consistent (i.e., has a solution), is tha the matrices A and [A B] are of the same rank.

Proof : Let $\rho(A) = r$ where A is m × n matrix.

$\therefore$ column rank of A is also r.

$\therefore$ r columns of A are linearly independent and the remaining (n – r) are linearly dependent.

Let $C_1, C_2, \dots, C_r$ be linearly independent and $C_{r+1}, \dots, C_n$ be linearly dependent where A = $[C_1 \quad C_2 \dots C_n]$.

The given equation is Ax = B where $X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$

$$\text{i.e., } \begin{bmatrix} C_1 & C_2 \dots C_r & C_{r+1} \dots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_r \\ x_{r+1} \\ : \\ x_n \end{bmatrix} = B$$

$$\text{i.e., } x_1C_1 + x_2C_2 + \dots + x_rC_r + x_{r+1}C_{r+1} + \dots + x_nC_n = B$$

Condition is necessary.

Asume that the equation AX = B has a solution $X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$

$$\therefore \text{ we have } \begin{bmatrix} C_1 & C_2 .. C_r & C_{r+1} \dots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_r \\ x_{r+1} \\ : \\ x_n \end{bmatrix} = B$$

$\therefore$ $x_1C_1 + x_2C_2 + \ldots + x_rC_r + x_{r+1}C_{r+1} + \ldots + x_nC_n = B$ … (2)

$\because$ $C_{r+1}\ C_{r+2}, \ldots, C_n$ are linearly dependent and

$C_1, C_2, .., C_r$ are linearly independent.

$\therefore$ $C_{r+1}, \ldots, C_n$ are linear combination of $C_1, C_2, \ldots, C_r$ and consequently from (2), B is also a linear combination of $C_1, C_2, \ldots, C_r$.

$\therefore$ number of linearly independent columns of [A B] is also r.

$\therefore$ if the equation AX = B has a solution, then rank of A is the same as that of [A B].

Condition is sufficient.

Assume rank of A as well as of [A B] is r.

$\because$ rank of [A B] is r.

$\therefore$ number of independent columns of [A B]

i.e., $\left[C_1 \quad C_2 \ldots C_r \quad C_{r+1} \ldots C_n \quad B\right]$ is r.

But C_1, C_2, …, C_r are already linearly independent.

$\therefore$ B is linearly dependent column

$\because$ B is a linear combination of C_1, C_2, …., C_r

$\therefore$ there exists r scalars p_1, p_2, …., p_r such that

$$B = p_1C_1 + p_2C_2 + \ldots + p_rC_r$$

The above equation can be written as

$$p_1C_1 + p_2C_2 + \ldots + p_rC_r + 0 \,.\, C_{r+1} + \ldots + 0.C_n = B \quad \ldots (3)$$

Comparing (1) and (3), we get,

$$x_1 = p_1, x_2 = p_2, \ldots, x_r = p_r, x_{r+1} = \ldots = x_n = 0.$$

$\therefore$ $X = \begin{bmatrix} p_1 \\ p_2 \\ : \\ p_r \\ 0 \\ : \\ 0 \end{bmatrix}$ is a solution of AX = B.

$\therefore$ if ranks of A and [A B] are same, the equation AX = B has a solution.

Article 3 : The equation AX = B has a unique solution if A is non-singular.

Proof : (i) Assume that A is non-singular i.e., A^{-1} exists.

$\therefore$ from the equation AX = B, we have,

$A^{-1}(AX) = A^{-1}B$ i.e., $X = A^{-1}B$ which is a solution of AX = B.

(ii) We prove that the solution is unique.

If possible, let X_1, X_2 be two different solutions of AX = B

$\therefore$ $AX_1 = B$ and $AX_2 = B$

Consequently $AX_1 = AX_2$

$\Rightarrow$ $A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Rightarrow$ $X_1 = X_2$

which is not possible as X_1, X_2 are distinct.

$\therefore$ our supposition is wrong.

$\therefore$ Ax = B has a unique solution.

2.4.5 Problem

Problem 1 : Find the value of k so that the equation

x – 2y + z = 0, 3x – ty + 2z = 0, y + kx = 0 have

(i) unique solution

(ii) infinitely many solution. Also find solutions for these values of k.

Solution : The given equations ar

x – 2y + z = 0

3x – y + 2 = 0

0x + y + kz = 0

which can be written as

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ AX = O

Where $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 1 & k \end{bmatrix}, \text{by } R_2 \to R_2 - 3R_1$$

$$= 1\begin{vmatrix} 5 & -1 \\ 1 & k \end{vmatrix} = 1(5k+1) = 5k+1$$

(i) Equations have a unique solution

if $|A| \neq 0$

i.e. if 5k + 1 ≠ 0

i.e. if $k \neq -\frac{1}{5}$

(ii)System has infinitely many solutions

if $|A| = 0$

i.e. if 5k + 1 = 0

i.e. if $k = -\frac{1}{5}$

When $k = -\frac{1}{5}$, we have

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_2 \to R_2 - 3R_1$$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to 5R_3$$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to R_3 - R_2$$

$\therefore \quad x - 2y + z = 0$

$$5y - z = 0 \Rightarrow 5y = z \Rightarrow y = \frac{1}{5}z$$

$$\therefore \quad x - \frac{2}{5}z + z = 0 \Rightarrow x + \frac{3}{5}z = 0 \Rightarrow x = -\frac{3}{5}z$$

Put z = k

$\therefore$ solutions are $x = -\frac{3}{5}k, y = \frac{1}{5}k, z = k$, where k is a parameter.

Problem 2 : Find non-trivial solution of the system of equations

$$x - 2y - 3z = 0$$

$$-2x + 3y + 5z = 0$$

$$3x + y - 2z = 0, \text{ if possible.}$$

Sol. The given equations are

$$x - 2y - 3z = 0$$

$$-2x + 3y + 5z = 0$$

$$3x + y - 2z = 0$$

which can be written as

$$\begin{bmatrix} 1 & -2 & -3 \\ -2 & 3 & 5 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 7 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 + 2R_1, R_3 \to R_3 - 3R_1$$

$$\therefore \quad \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + 7R_2$$

$$\therefore \quad x - 2y - 3z = 0$$

$$-y - z = 0 \quad \Rightarrow y = -z$$

$$\therefore \quad x + 2z - 3z = 0 \;\Rightarrow x = z$$

Put z = k

$\therefore$ x = k, y = –k, z = k, where k is a parameter.

Problem 3 : Show that the system of equations

$$x + y + z = 4, 2x + 5y - 2z = 3, x + 7y - 7z = -6$$

is consistent and solve it.

Sol. The given equations are

$$x + y + z = 4$$

$$2x + 5y - 2z = 3$$

$$x + 7y - 7z = -6$$

which can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 6 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -10 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - 2R_2$$

Now rank of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ as well as of $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is 2 .

$\therefore$ given equations are consistent and solutions are given by

$$x + y + z = 4.$$

$$3y - 4z = -5 \Rightarrow 3y = 4z - 5 \Rightarrow y = \frac{4}{3}z - \frac{5}{3}$$

$$\therefore \quad x + \frac{4}{3}x - \frac{5}{3} + z = 4 \Rightarrow x + \frac{7}{3}z = \frac{17}{3} \Rightarrow x = -\frac{7}{3}z + \frac{17}{3}$$

Put z = k

$\therefore$ solutions are $x = -\frac{7}{3}k + \frac{17}{3}, y = \frac{4}{3}k - \frac{5}{3}, z = k$

where k is a parameter.

Problem 4 : Investigate for what values of a, b the following equations

$$x + y + 5z = 6$$

$$x + 2y + 3az = b$$

$$x + 3y + ax = 1$$

have

1. no solution
2. unique solution
3. an infinite number of solutions.

Sol. The given equations are

$$x + y + 5z = 6$$

$$x + 2y + 3az = b$$

$$x + 3y + ax = 1$$

which can be written as

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ b \\ 1 \end{bmatrix}$$

i.e., $AX = B$ where $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ b \\ 1 \end{bmatrix}$

The given equations will have a unique solution.

if $\begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{vmatrix} \neq 0$

i.e., if $\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 2 & a-5 \end{vmatrix} \neq 0$, by $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

i.e., if $\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 0 & -5a+5 \end{vmatrix} \neq 0$, by $R_3 \to R_3 - 2R_2$

i.e., if $-5a + 5 \neq 0$ i.e., if $a \neq 1$

$\therefore$ given equations will have a unique solution when $a \neq 1$ and b has any value When $a = 1$,

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - 2R_2$$

$\therefore \rho(A) = 2$

$$[A\ B] = \begin{bmatrix} 1 & 1 & 5 & 6 \\ 1 & 2 & 3 & b \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 1 & -2 & b-6 \\ 0 & 2 & -4 & -5 \end{bmatrix}, \text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 1 & -2 & b-6 \\ 0 & 0 & 0 & -2b+7 \end{bmatrix}, \text{by } R_3 \to R_3 - 2R_2$$

Rank of [A B] is 3 if $b \neq \frac{7}{2}$

$\therefore$ rank of A and [A B] are not equal if $b \neq \frac{7}{2}$

$\therefore$ if a = 1, $b \neq \frac{7}{2}$, the given set of equations does not have any solution. If $a = 1, b = \frac{7}{2}$, then the ranks of A and [A B] are equal and A is singular.

$\therefore$ the given system of equations has an infinite number of solutions.

Self Check Exercise

1. Show that the equations

$$x + y + z + 3 = 0$$

$$3x + y - 2z + 2 = 0$$

$$2x + 4y + 7z - 7 = 0$$

are inconsistent.

....

....

2.4.6 Summary

In this lesson, we have studied about the homogeneous AX = 0 and non-homogeneous system of linear equations AX = B and their solutions. We have studied various conditions under which a system can possess different types of solutions such as unique solution, no solutions and infinite many solutions. Moreover, more clarity of concept has been developed by using some simple examples.

2.4.7 Key Concepts

System of homogeneous linear equation, System of non-homogeneous linear equations, Linearly independent solutions, Consistency, Unique solution, No solution, Infinite many solutions,

2.4.8 Long Questions

1. Determine the value of λ so that the equations

$$2x + y + 2z = 0$$

$$x + y + 3z = 0$$

$$4x + 3y + \lambda z = 0$$

have non-zero solution.

2. Solve the following equations :

$$x + y + z = 0$$

$$x + 2y + 3z = 0$$

$$x + 3y + 4z = 0$$

2.4.9 Short Questions

1. For what value of λ, does the system $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$ has (i) a unique solution (ii) more than one solution.

2. Solve the equations

$$x - y + z = 5$$

$$2x + y - z = -2$$

$$3x - y - z = -7$$

3. Examine the consistency of the following equations and if consistent, find the complete solution

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 9z = 21$$

2.4.10 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.