Centre for Distance and Online Education

Punjabi University, Patiala

Class : B.A. I (Mathematics) **Semester : I**

Paper : MTHB1101T **Unit-2**

(Calculus)

Medium : English

Lesson No.

Website : www.pbidde.org

MTHB1101T: CALCULUS

Course Outcomes:	
CO1	To understand the order completeness properties of real numbers
CO2	Able to learn basic properties of limits, infinite limits, indeterminate forms.
CO3	To understand Continuous functions, types of discontinuities, continuity of composite functions.
CO4	To know Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, their geometric interpretation and applications.
CO5	To understand Hyperbolic, inverse hyperbolic functions of a real variable and their derivatives.

For Regular Students / Students of Centre for Distance and Online Education
Maximum Marks: 50 Marks

Maximum Time: 3 Hrs.

For Regular students:6Lectures of 45minutes/week

External Marks: 35
Internal Assessment: 15
Pass Percentage: 35%
For Private Students
Maximum Marks: 50 Marks

INSTRUCTIONS FOR THE PAPER-SETTER

The question paper will consist of three sections A, B and C. Sections A and B will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eleven short answer type questions covering the entire syllabus uniformly. Each question in Sections A and B will be of 06 marks and Section C will be of 11 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Sections A and B and compulsory question of Section C.

SECTION-A

Properties of real numbers :
Order property of real numbers, bounds, l.u.b. and g.l.b. order completeness property of real numbers, archimedian property of real numbers.
Limits: ε -δ definition of the limit of a function, basic properties of limits, infinite limits, indeterminate forms.
Continuity: Continuous functions, types of discontinuities, continuity of composite functions, continuity of f (x) , sign of a function in a neighborhood of a point of continuity, intermediate value theorem, maximum and minimum value theorem.

Head,
Department of Mathematics
Punjabi University, Patiala

SECTION-B

Mean value theorems: Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, their geometric interpretation and applications, Taylor's theorem, Maclaurin's theorem with various form of remainders and their applications.

Hyperbolic, inverse hyperbolic functions of a real variable and their derivatives, successive differentiations, Leibnitz's theorem.

REFERENCES :

1. J. D. Murray & M . R. Spiegel : Theory and Problems of Advanced Calculus, Schaum's Outline Series, Schaum Publishing Co., New York.

2. P.K. Jain and S. K. Kaushik : An Introduction to Real Analysis, S. Chand & Co., New Delhi, 2000.

3. Gorakh Prasad : Differential Calculus, Pothishala Private Ltd., Allahabad.

4. G.B. Thomas & R.L. Finney : Calculus and Analytic Geometry (Ninth Edition), Pearson Publication.

5.Shanti Narayan and P.K. Mittal : Differential Calculus, Edition 2006, S. Chand & Co., New Delhi.

Head,
Department of Mathematics

Updated on April, 2024

B.A. PART - I **MTHB1101T**
(SEMESTER-I) **CALCULUS**

LESSON NO. 2.1 **Author : Dr. Chanchal**

SUCCESSIVE DIFFERENTIATION -I

2.1.1 Objectives

During the study in this particular lesson, our main objectives are

* To obtain n^{th} order derivatives of some standard functions by the method of mathematical induction.

2.1.2 Introduction

We are already familiar with the concept that derivative of a function of x is also a function of x. Thus the derivative of a function may have its derivative without any loss of genrality.

If y = f(x),

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$$

is called the first differential coefficient or first derivative of f(x). If the process of differentiation be continued in succession, we obtain second, third and higher order derivatives, as follows :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\delta x \to 0} \frac{f'(x+\delta x) - f'(x)}{\delta x} = f''(x),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \lim_{\delta x \to 0} \frac{f''(x+\delta x) - f''(x)}{\delta x} = f'''(x)$$

and so on. These are also denoted by

$$y_1 = \frac{dy}{dx} = Dy,\ y_2 = \frac{d^2y}{dx^2} = D^2, \ldots, y_n = \frac{d^ny}{dx^n} = D^ny$$

2.1.3 Successive Differentiation of Some Standard Functions

Art 1 : Prove the following results :

(i) If $y = (ax + b)^m$, then $y_n = m(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}a^n$.

Proof : Here $y = (ax + b)^m$

Differentiating both sides w.r. D^2y we get,

$y_1 = m(ax + b)^{m-1}.\ a = m(ax + b)^{m-1}.\ a^1$

∴ result is true for n = 1

Assume that the result is true for n = k, where k is positive integer.

∴ $y_k = m(m-1)(m-2)\ldots(m-k+1)(ax+b)^{m-k}.\ a^k$

Differentiating both sides w.r.t.x, we get,

$y_{k+1} = m(m-1)(m-2)\ldots(m-k+1)(m-k)(ax+b)^{m-k-1}.\ a\ .\ a^k$

or $y_{k+1} = m(m-1)(m-2)\ldots(m-k+1)(m-k)(ax+b)^{m+(k+1)}.\ a^{k+1}$

∴ result is true for n = k + 1.

∴ if the result is true for any positive integer k, then it is also true for the next higher integer k + 1.

But the result is true for n = 1 also.

∴ By method of induction, the result is true for all positive integers n.

Cor. I. If m is a positive integer > n, then

$$y_n = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)\underline{|m-n}}{\underline{|m-n}}(ax+b)^{m-n}.\ a^n$$

or $$y_n = \frac{\underline{|m}}{\underline{|m-n}}(ax+b)^{m-n}.\ a^n$$

If m = n, then $y_n = n(n-1)(n-2)\ldots.\ 2.1\ (ax+b)^0.\ a^n$

or $$y_n = \underline{|n}\ .\ a^n$$

$y_{n+1} = y_{n+2} = \ldots = 0$

∴ $y_n = 0\ \forall\ n > m$.

(ii) If $y = \dfrac{1}{ax+b}$, then $y_n = \dfrac{(-1)^n\ \underline{|n}\ .\ a^n}{(ax+b)^{n+1}},\ x \neq -\dfrac{b}{a}$.

Proof : Here $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$\therefore$ $$y_1 = (-1)(ax+b)^{-2} \cdot a = \frac{(-1)^{-1} \cdot \underline{|1} \cdot a^1}{(ax+b)^2}$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$\therefore$ $$y_k = \frac{(-1)^k \cdot \underline{|k} \cdot a^k}{(ax+b)^{k+1}} = (-1)^k \underline{|k}\, a^k (ax+b)^{-k-1}$$

Differentiating again w.r.t. x, we get,

$$y_{k+1} = (-1)^k \underline{|k}\, a^k (-k-1)(ax+b)^{-k-2} \cdot a$$

$$= (-1)^{k+1} \underline{|k+1}\, a^{k+1} (ax+b)^{-(k+2)} = \frac{(-1)^{k+1} \underline{|k+1} \cdot a^{k+1}}{(ax+b)^{k+2}}$$

$\therefore$ result is true for n = k + 1.

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By method of induction, the result is true for all positive integers n.

(iii) If y = log (ax + b), then $y_n = \dfrac{(-1)^{n-1} \underline{|n-1} \cdot a^n}{(ax+b)^n}, x > -\dfrac{b}{a}$.

Proof : Here y = log (ax + b)

Differentiating both sides w.r.t.x,

$$y_1 = \frac{1}{ax+b} \cdot a = \frac{(-1)^{1-1} \cdot \underline{|1-1} \cdot a^1}{(ax+b)^1}$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$\therefore$ $$y_k = \frac{(-1)^{k-1} \cdot \underline{|k-1} \cdot a^k}{(ax+b)^k} = (-1)^{k-1} \underline{|k-1} \cdot a^k \cdot (ax+b)^{-k}$$

Differentiating both sides w.r.t.x, we get,

$$y_{k+1} = (-1)^{k-1} \underline{|k-1}\, a^k (-k)(ax+b)^{-k-1} \cdot a$$

$$= (-1)^k \underline{|k}\, a^{k+1} (ax+b)^{-(k+1)} = \frac{(-1)^k \underline{|k} \cdot a^{k+1}}{(ax+b)^{k+1}}$$

$\therefore$ result is true for n = k + 1

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1

But the result is true for n = 1.

$\therefore$ By the method of induction, the result is true for all positive integers n.

Note : Same result will hold even if $y = \log |ax + b|$ where $x > -\dfrac{b}{a}$.

(iv) If $y = a^{mx}$, a > 0, then $y_n = a^{mx}. (\log a)^n . m^n$.

Proof : Here $y = a^{mx}$

Differentiating both sides w.r.t.x,

$y_1 = a^{mx}, \log a . m = a^{mx}. (\log a)^1 . m^1$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$\therefore$ $y_k = a^{mx} . (\log a)^k . m^k$

Differentiating both sides w.r.t.x,

$y_{k+1} = [a^{mx} . (\log a) . (m)] . (\log a)^k . m^k = a^{mx} . (\log a)^{k+1} . m^{k+1}$

$\therefore$ result is true for n = k + 1.

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By the method of induction, the result is true for all positive integers n.

Cor. 1. Put m = 1

$\therefore$ $y_n = a^x . (\log a)^n$

$\therefore$ $y = a^x \Rightarrow y_n = a^x . (\log a)^n$

Cor. 2. Put a = e

$\therefore$ $y_n = e^{mx} . (\log e)^n . m^n = e^{mx} . m^n$

$\therefore$ $y = e^{mx} \Rightarrow y_n = e^{mx} . m^n$

Cor. 3. Put a = e, m = 1

$\therefore$ $y_n = e^x . (\log e)^n . (1)^n = e^x$

$\therefore$ $y = e^x \Rightarrow y_n = e^x$.

(v) If y = sin (ax + b), then $y_n = a^n \sin\left(ax + b + \dfrac{n\pi}{2}\right) \forall x \in R$.

Proof : Here y = sin (ax + b)

Differentiating both sides w.r.t. x,

$$y_1 = \cos(ax + b) . a = a^1 \sin\left[ax + b + 1 . \frac{\pi}{2}\right]$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$\therefore \quad y_k = a^k \sin\left[ax + b + k\frac{\pi}{2}\right]$

Differentiating again w.r.t.x,

$$y_{k+1} = a^k \cos\left[ax + b + k\frac{\pi}{2}\right].a = a^{k+1}\sin\left[\left(ax + b + k\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= a^{k+1}\sin\left[ax + b + (k+1)\frac{\pi}{2}\right]$$

$\therefore$ result is true for n = k + 1.

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By the method of induction, the result is true for all positive integers n.

(vi) If y = cos (ax + b), then $y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right) \forall\, x \in R$.

Proof : The proof is left as an exercise for the reader.

(vii) If $y = e^{ax}\sin(bx + c)$, then $y_n = (a^2 + b^2)^{\frac{\pi}{2}} e^{ax} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

Proof : Here $y = e^{ax}\sin(bx + c)$

Differentiating both sides w.r.t.x,

$$y_1 = e^{ax}.\frac{d}{dx}[\sin(bx + c)] + \sin(bx + c).\frac{d}{dx}(e^{ax})$$

$= e^{ax}.\cos(bx + c).b + \sin(bx + c).e^{ax}.a$

$\therefore \quad y_1 = e^{ax}[a\sin(bx + c) + b\cos(bx + c)]$

Put $a = r\cos\alpha$ and $b = r\sin\alpha$ where $r > 0$.

Squaring and adding (2) and (3), we get,

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

Dividing (3) by (2), tan $\alpha = \frac{b}{a} \Rightarrow a = \tan^{-1}\frac{b}{a}$

$\therefore$ from (1), $y_1 = e^{ax}[r\cos\alpha\sin(bx + c) + r\sin\alpha\cos(bx + c)]$

$= e^{ax}.r[\sin(bx + c)\cos\alpha + \cos(bx + c).\sin\alpha]$

$= r\,e^{ax}\sin(bx + c + \alpha)$

$\therefore \quad y_1 = (a^2 + b^2)^{\frac{1}{2}} \,.\, e^{ax} \sin\left(bx + c + 1 \,.\, \tan^{-1}\frac{b}{a}\right)$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer

$\therefore \quad y_k = (a^2 + b^2)^{\frac{k}{2}} \,.\, e^{ax} \sin\left(bx + c + k \tan^{-1}\frac{b}{a}\right)$

or $y_k = r^k e^{ax} \sin(bx + c + k\alpha)$

Differentiating again w.r.t. x, we get,

$y_{k+1} = r^k \,.\, [e^{ax} \,.\, \cos(bx + c + k\alpha) \,.\, b + \sin(bx + c + k\alpha) \,.\, ae^{ax}]$

$= r^k \,.\, e^{ax} [a \sin(bx + c + k\alpha) + b \cos(bx + c + k\alpha)]$

$= r^k \,.\, e^{ax} [r \cos\alpha \sin(bx + c + k\alpha) + r \sin\alpha \cos(bx + c + k\alpha)]$

$= r^{k+1} \,.\, e^{ax} \sin[(bx + c + k\alpha) + \alpha] = r^{k+1} \,.\, e^{ax} \sin[bx + c + (k + 1)\alpha]$

$\therefore \quad y_{k+1} = (a^2 + b^2)^{\frac{k+1}{2}} \,.\, e^{ax} \sin\left(bx + c + (k+1) \tan^{-1}\frac{b}{a}\right)$

$\therefore$ the result is true for n = k + 1

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By the method of induction, the result is true for all positive integers.

(viii) If $y = e^{ax} \cos(bx + c)$, then

$$y_n = (a^2 + b^2)^{\frac{n}{2}} \,.\, e^{ax} \cos\left(bx + c + n \tan^{-1}\frac{b}{a}\right)$$

Proof : The proof is left as an exercise for the reader.

2.1.4 Some Important Examples

Example 1 : If y = cosh (log x) + sinh (log x), prove that $y_n = 0$ for n > 1.

Sol. y = cosh (log x) + sinh (log x)

Differentiating both sides w.r.t.x, we get

$$y_1 = \sinh(\log x) \,.\, \frac{1}{x} + \cosh(\log x) \,.\, \frac{1}{x}$$

$\therefore \quad xy_1 = \sinh(\log x) + \cosh(\log x)$

or $xy_1 = y$

Again differentiating w.r.t.x, we get

$xy_2 + y_1 = y_1$ or $xy_2 = 0$

$\therefore \quad y_2 = 0$

$\therefore \quad y_n = 0$ for x > 1.

Example 2 : If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

Sol. Here $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$\Rightarrow \quad p^2 = a^2 \left(\frac{1+\cos 2\theta}{2}\right) + b^2 \left(\frac{1-\cos 2\theta}{2}\right)$

$\Rightarrow \quad 2p^2 = a^2 (1 + \cos 2\theta) + b^2 (1 - \cos 2\theta)$

$\Rightarrow \quad 2p^2 - (a^2 + b^2) = (a^2 - b^2) \cos 2\theta$ ... (1)

Differentiating w.r.t. θ, we get,

$$4p \frac{dp}{d\theta} = -2 (a^2 - b^2) \sin 2\theta$$

or $\quad -2p \frac{dp}{d\theta} = (a^2 - b^2) \sin 2\theta$ ... (2)

Squaring (1), (2) and adding, we get,

$$4p^4 + (a^2 + b^2)^2 - 4p^2 (a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 = (a^2 - b^2)^2$$

or $\quad 4p^4 - 4p^2(a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 + (a^2 + b^2)^2 - (a^2 - b^2)^2 = 0$

or $\quad 4p^4 - 4p^2(a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 + 4a^2b^2 = 0$

Dividing both sides by $4p^2$, we get,

$$p^2 - (a^2 + b^2) + \left(\frac{dp}{d\theta}\right)^2 + \frac{a^2b^2}{p^2} = 0$$

Dividing by $2\frac{dp}{d\theta}$, we get,

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Example 3 : Find the nth derivative of $\sqrt{ax+b}$.

Sol. Let $y = \sqrt{ax+b} = (ax+b)^{\frac{1}{2}}$

$$\therefore \quad y_1 = \frac{1}{2}(ax+b)^{\frac{1}{2}-1} . a = \frac{1}{2}(ax+b)^{-\frac{1}{2}} a$$

$$\therefore \quad y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)(ax+b)^{-\frac{3}{2}} a^2 = y_3 = \frac{(-1)^2\, 1.3}{2}(ax+b)^{\frac{1}{2}-3} a^3$$

$$\therefore \quad y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right).(ax+b)^{-\frac{5}{2}} a^3$$

$(-1)^1.1$ 2 2

2^2

$$\therefore \quad y_3 = \frac{(-1)^2\, 1.3}{2}(ax+b)^{\frac{1}{2}-3} a^3$$

....

$$\therefore \quad y_n = \frac{(-1)^{n-1}\, 1.3.5\ldots(2n-1)\,(ax+b)^{\frac{1}{2}-n} a^n}{2^n}$$

3

$$\therefore \quad y_n = \frac{(-1)^{n-1} 1.3.5\ldots(2n-1)}{2^n (ax+b)^{\frac{2n-1}{2}}} . a^n \text{ where } x \neq -\frac{b}{a}$$

2

Example 4 : Find y_n if $y = \frac{2x+1}{(x-2)(x-1)^3}$.

Sol. $y = \frac{2x+1}{(x-2)(x-1)^3}$

Put $\frac{2x+1}{(x-2)(x-1)^3} \equiv \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$

Multiplying both sides by $(x-2)(x-1)^3$, we get

$2x + 1 \equiv A(x-1)^3 + B(x-2)(x-1)^2 + C(x-2)(x-1) + D(x-2)$ ... (1)

Putting $x - 2 = 0$ i.e. $x = 2$ in (1), we get

$5 = A \Rightarrow A = 5$

Putting $x - 1 = 0$ i.e. $x = 1$ in (1), we get

$3 = -D \Rightarrow D = -3$

(1) can be writing as

$$2x + 1 = A\,(x^3 - 3x^2 + 3x - 1) + B\,(x^3 - 4x^2 + 5x - 2) + C\,(x^2 - 3x + 2) + D\,(x - 2) \quad \ldots (2)$$

Equating coefficients in (2) of

x^3) $A + B = 0 \Rightarrow 5 + B = 0 \Rightarrow B = -5$

x^2) $-3A - 4B + C = 0 \Rightarrow -15 + 20 + C = 0 \Rightarrow C = -5$

$$\therefore \quad \frac{2x+1}{(x-2)\,(x-1)^2} \equiv \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y = \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y_n = 5\,\frac{(-1)^n \underline{|n}}{(x-2)^{n+1}} - 5\,\frac{(-1)^n \underline{|n}}{(x-1)^{n+1}} - 5\,\frac{(-1)^n \underline{|n+1}}{(x-2)^{n+2}} - 3\,\frac{(-1)^n \underline{|n+2}}{(x-1)^{n+3}}$$

$$\therefore \quad y_n = (-1)^n\,\underline{|n}\,\frac{5}{(x-2)^{n+1}} - \frac{5}{(x-1)^{n+1}} - \frac{5(n+1)}{(x-1)^{n+2}} - \frac{3(n+2)\,(n+1)}{(x-1)^{n+3}}$$

Example 5 : Find the nth derivative of $y = e^{3x} \sin^2 2x$.

Sol. $y = e^{3x} \sin^2 2x = e^{3x}\,\dfrac{1-\cos 4x}{2} = \dfrac{1}{2} e^{3x} - \dfrac{1}{2} e^{3x} \cos 4x$

$$\therefore \quad y_n = \frac{1}{2} e^{3x} \,.\, 3^n - \frac{1}{2} (9+16)^{\frac{n}{2}}\; e^{3x} \cos 4x + n \tan^{-1} \frac{4}{3}$$

$$\therefore \quad y_n = \frac{1}{2}\, e^{3x}\; 3^n - 5^n \cos 4x + n \tan^{-1} \frac{4}{3}\,.$$

Self Check Exercise

1. Find the n^{th} derivative of : sin x sin 2x

....

....

....

2.1.5 Summary

In this lesson, we have explained the concept of successive differentiation of some standard functions by using the principle of mathematical induction. The topic is made more clear with the help of several simple examples.

2.1.6 Key Concepts

Higher order derivatives, Successive differentiation.

2.1.7 Long Questions

1. If $y = e^{ax} \sinh bx$ prove that $y_2 - 2ay_1 + (a^2 - b^2)\,y = 0$.
2. If $y = \log(1 + \cos x)$, prove that $y_1 y_2 + y_3 = 0$.
3. If $x = \sin\theta$, $y = \sin m\theta$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2 y = 0$.
4. If $y = \sin(m \sin^{-1}x)$, prove that

   $$(1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - (n^2 - m^2)\,y_n = 0$$
5. If $x = \tan(\log y)$, prove that

   $(1 + x^2)\,y_{n+2} + \{2(n + 1)\,x - 1\}\,y_{n+1} + n(n + 1)\,y_n = 0$.

2.1.8 Short Questions

1. Find the n^{th} derivative of ": $e^x \cos x \cos 2x$

2.1.9 Suggested Readings

1.	Ahsan Akhtar & Sabita Ahsan	:	Differential Calculus
2.	UP Singh, RJ Srivastava & NH Siddiqui	:	Differential Calculus
3.	Gorakh Prasad	:	Differential Calculus

LESSON NO. 2.2 **Author : Dr. Chanchal**

SUCCESSIVE DIFFERENTIATION -II

2.2.1 Objectives

The prime objectives of this lesson is to :

* To discuss Leibnitz's theorem for finding the n^{th} order derivatives of the product of two functions.
* To introduce some basic formulae related to the differentiation of hyperbolic and inverse hyperbolic functions.

2.2.2 Leibnitz's Theorem

Statement : If u and v are functions of x possessing nth order derivatives, then

$$(uv)_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \dots + {}^nC_r\, u_{n-r} v_r + \dots + {}^nC_n\, uv_n$$

where u_r denotes the rth order derivative of u and ${}^nC_r$ denotes the number of combinations out of n different things taken r at a time.

Proof : We have

$$(uv)_1 = u_1 v + uv_1 = {}^1C_0\, u_1 v + {}^1C_1\, uv_1$$

$\therefore$ theorem is true for n = 1.

Assume that the theorem is true for n = m, where m is a positive integer.

$$\therefore \quad (uv)_m = {}^mC_0\, u_m v + {}^mC_1\, u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots + {}^mC_{r-1}\, u_{m-r+1} v_{r-1} + {}^mC_r\, u_{m-r} v_r + \dots + {}^mC_m\, uv_m$$

Differentiating both sides w.r.t.x, we get,

$$(uv)_{m+1} = {}^mC_0\, u_{m+1} v + {}^mC_0\, u_m v_1 + {}^mC_1\, u_m v_1 + {}^mC_1\, u_{m-1} v_2 + \dots \dots \dots \dots \dots$$

$+ \ {}^mC_{r-1}\, u_{m-r+2}\, v_{r-1} + {}^mC_{r-1}\, u_{m-r+1}\, v_r$

$+ \ {}^mC_r\, u_{m-r+1}\, v_r + {}^mC_r\, u_{m-r}\, v_{r+1}$

$+ \ldots \ \ldots \ \ldots \ \ldots \ldots$

$+ \ {}^mC_m\, u_1\, v_m + {}^mC_m\, uv_{m+1}$

$\therefore \ (uv)_{m+1} = {}^mC_0\, u_{m+1}\, v + ({}^mC_0 + {}^mC_1)\, u_m\, v_{1+} ({}^mC_1 + {}^mC_2)\, u_{m-1}\, v_2$

$+ \ldots + ({}^mC_{r-1} + {}^mC_r)\, u_{m-r+1}\, v_r + \ldots + {}^mC_m\, uv_{m+1}$

But $\quad {}^mC_0 = 1 = {}^{m+1}C_0$

${}^mC_0 + {}^mC_1 = {}^{m+1}C_1$

${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$

$\ldots \ \ldots \ \ldots \ \ldots \ \ldots$

${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

${}^mC_m = 1 = {}^{m+1}C_{m+1}$

$\therefore$ we have

$(uv)_{m+1}\ {}^{m+1}C_0\, u_{m+1}\, v + {}^{m+1}C_1 u_m\, v_1 + {}^{m+1}C_2\, u_{m-1}\, v_2 + \ldots$

$+ \ {}^{m+1}C_r\, u_{m-r+1}\, v_r + \ldots + {}^{m+1}C_{m+1}\, uv_{m+1}$

$\therefore$ theorem is true for n = m + 1.

$\therefore$ if the theorem is true for n = m, then it is also true for n = m + 1

But the theorem is true for n = 1.

$\therefore$ By the method of induction, theorem is true for all positive integers n.

2.2.2.1 Some Important Examples

Example 1 : Prove that $\dfrac{d^n}{dx^n}\left[\dfrac{\log x}{x}\right] = \dfrac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n}\right]$

Given that x > 0.

Sol. Here $y = \dfrac{\log x}{x} = \log x \cdot \dfrac{1}{x}$

Let V = log x $\qquad U = \dfrac{1}{x}$

$V_1 = \dfrac{1}{x} = x^{-1} \qquad U_1 = (-1)\, x^{-2}$

$V_2 = (-1)\, x^{-2} \qquad U_2 = (-1)\,(-2)\, x^{-3}$

$V_3 = (-1)\,(-2)\, x^{-3} \qquad U_3 = \dfrac{(-1)^3 \lfloor 3}{x^4}$

and so on $\qquad$ and so on

$$V_n = \frac{(-1)^{n-1}\,\underline{|n-1}}{x^n} \qquad\qquad U_n = \frac{(-1)^n\,\underline{|n}}{x^{n-1}}$$

By Leibnitz's rule

$$\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}(U\,.\,V) = \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)$$

$$= {}^nC_0\,\frac{(-1)^n\,\underline{|n}}{x^{n+1}}\,.\log x + {}^nC_1\,\frac{(-1)^{n-1}\underline{|n-1}}{x^n}\,.\,\frac{1}{x}$$

$$= {}^nC_2\,\frac{(-1)^{n-2}\,\underline{|n-2}}{x^{n+1}}\,.\,\frac{(-1)}{x^2} + \ldots. + {}^nC_n\,.\,\frac{1}{x}\,\frac{(-1)^{n-1}\underline{|n-1}}{x^n}$$

$$= \frac{(-1)^n\underline{|n}}{x^{n-1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right].$$

Example 2 : If $y = (\sin^{-1} x)^2$, find $y_n(0)$.

Sol. $y = (\sin^{-1} x)^2$

Differentiating w.r.t. x,

$$y_1 = 2\,(\sin^{-1} x)\,.\,\frac{1}{\sqrt{1-x^2}}$$

Squaring and cross-multiplying

$(1 - x^2)\, y_1^2 = 4\,(\sin^{-1} x)^2 \qquad \Rightarrow (1 - x^2)\, y_1^2 = 4y$

$\therefore \qquad (1 - x^2)\, y_1^2 - 4y = 0$

Differentiating w.r.t.x, again we get,

$(1 - x^2)\, 2y_1 y_2 - 2x\, y_1^2 - 4y_1 = 0$

Dividing by $2y_1$, we get

$(1 - x^2)\, y_2 - xy_1 - 2 = 0$ … (3)

Differentiating n times (3) by Leibnitz's rule,

$$1.\; y_{n+2}\,(1-x^2) + \frac{n}{1}\,y_{n+1}\,(-2x) + \frac{n(n-1)}{2.1}\,y_n(-2) - 1\,.\,y_{n+1}\,x - \frac{n}{1}\,y_n\,.\,1 - 0 = 0$$

$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$ … (4)

Putting x = 0 in (1), (2), (3) and (4) we get,

$y\,(0) = 0$ … (5)

$y_1(0) = 0$ ... (6)

$y_2(0) = 2$ ... (7)

$y_{n+2} = n^2 y_n(0)$ ... (8)

Putting n = 1, 2, 3, 4 ... in (8), we get,

$y_3(0) = 1^2 y_1(0) = 0$ ... (9) $[\because \text{ of } (6)]$

$y_4(0) = 2^2 y_2(0) = 2.2^2$ ... (10) $[\because \text{ of } (7)]$

$y_5(0) = 3^2 y_3(0) = 0$ ... (11) $[\because \text{ of } (8)]$

$y_6(0) = 4^2 y_4(0) = 2 \, . \, 2^2 \, . \, 4^2$ $[\because \text{ of } (10)]$

and so on.

$\therefore$ In general $y_n(0) = \begin{cases} 2.2^2.4^2...(n-2)^2 & \text{when n is even and } n \neq 2 \\ 0 & \text{when n is odd} \end{cases}$

2.2.3 Some Important Formulae

1. $\sinh x = \dfrac{e^x - e^{-x}}{2}$

2. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

3. $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{\sinh x}{\cosh x}$

4. $\cosh x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{\cosh x}{\sinh x}$

5. $\text{sec h } x = \dfrac{2}{e^x + e^{-x}} = \dfrac{1}{\cosh x}$

6. $\text{cosech } x = \dfrac{2}{e^x - e^{-x}} = \dfrac{1}{\sinh x}$

7. $\sinh 0 = 0, \cosh 0 = 1$

8. $\cosh^2 x - \sinh^2 x = 1$

9. $\text{sech}^2 x = 1 - \tanh^2 x$

10. $\text{cosech}^2 x = \coth^2 x - 1$

11. $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2 \tanh x}{1-\tanh^2 x}$

12. $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$ $\qquad \cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$= 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$ $\qquad \sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

13. $\tanh\ 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

14. $\sinh (x + y) = \sinh x \cosh y + \cos x \sinh y$

15. $\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y$

16. $\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y$

17. $\cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y$

18. $\sinh (x + y) \sinh (x - y) - \sinh^2 x - \sinh^2 y$

19. $\cosh (x + y) \cosh (x - y) = \cosh^2 x + \sinh^2 y$

20. $2 \sinh x \cosh y = \sinh (x + y) + \sinh (x - y)$

21. $2 \cosh x \sinh y = \sinh (x + y) - \sinh (x - y)$

22. $2 \cosh x \cosh y = \cosh (x + y) + \cosh (x - y)$

23. $2 \sinh x \sinh y = \cosh (x + y) - \cosh (x - y)$

24. $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

25. $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

26. $\cosh x - \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

27. $\cosh x - \cosh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

28. $\dfrac{d}{dx} (\sinh x) = \cosh x$

29. $\frac{d}{dx}(\cosh x) = \sinh x$

30. $\frac{d}{dx}(\tanh x) = \text{sech}^2\, x$

31. $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

32. $\frac{d}{dx}(\text{sec h}\, x) = -\text{sech}\, x \tanh x$

33. $\frac{d}{dx}(\text{cosech}\, x) = -\ \text{cosech}\, x \coth x$

34. $\int \sinh x\, dx = \cosh x$

35. $\int \cosh dx = \sinh x$

36. $\int \text{sech}^2 x\, dx = \tanh x$

37. $\int \text{cosech}^2 x\, dx = -\coth x$

38. $\int \text{sech}\, x \tanh x\, dx = \text{-}\,\text{sech}\, x$

39. $\int \text{cosech}\, x \coth x\, dx = \text{-}\,\text{cosech}\, x$

40. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}, x \in R$

41. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$

42. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$

43. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1$

44. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{1}{|x|\sqrt{1-x^2}}, 0 < x < 1$

45. $\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{1}{|x|\sqrt{x^2+1}}, x \neq 0$

Self Check Exercise

1. Find the n^{th} derivative of : $x \log\left(\frac{x-a}{x+a}\right), x > a > 0$

....
....
....

2.2.4 Summary

In this lesson, we have stated and proved the Leibnitz's theorem. Moreover, several examples have been solved for a clear understanding of the concept. Further, we have introduced some basic formulae for the differentiation of hyperbolic and inverse hyperbolic functions.

2.2.5 Key Concepts

Leibnitz's theorem and its applications, Differentiation of hyperbolic and inverse hyperbolic functions.

2.2.6 Long Questions

1. If $y = x^n \log x$, prove that $y_{n+1} = \frac{\lfloor n}{x}$.
2. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0$.

2.2.7 Short Questions

1. Find the n^{th} derivative of : $2^x . e^x$

2.2.8 Suggested Readings

1. Ahsan Akhtar & Sabita Ahsan : Differential Calculus
2. UP Singh, RJ Srivastava & NH Siddiqui : Differential Calculus
3. Gorakh Prasad : Differential Calculus

Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.