Centre for Distance and Online Education

Punjabi University, Patiala

Class : B.A. I (Mathematics) **Semester : I**
Paper : MTHB1102T **Unit-I**
(Algebra and Trigonometry)
Medium : English

Lesson No.

1.1 : De-Moivre's Theorem and its Applications

1.2 : Functions of Comples Variables

1.3 : Summation of Series and Hyperbolic Functions

Department website : www.pbidde.org

MTHB1102T: ALGEBRA AND TRIGONOMETRY

Course Outcomes:	
CO1	To understand D'Moivre's theorem, application of D'Moivre's theorem.
CO2	To know about exponential, logarithmic, direct and inverse circular and hyperbolic functions of a complex variable.
CO3	To understand Summation of series including Gregory Series.
CO4	To know Hermitian and skew-hermitian matrices,linear dependence of row and column vectors.
CO5	To understand Eigen-values, eigen-vectors and characteristic equation of a matrix.

For Regular Students / Students of Centre for Distance and Online Education
Maximum Marks: 50 Marks

Maximum Time: 3 Hrs.
For Regular students:6Lectures of 45minutes/week

External Marks: 35
Internal Assessment: 15
Pass Percentage: 35%
For Private Students
Maximum Marks: 50 Marks

INSTRUCTIONS FOR THE PAPER-SETTER

The question paper will consist of three sections A, B and C. Sections A and B will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eleven short answer type questions covering the entire syllabus uniformly. Each question in Sections A and B will be of 06 marks and Section C will be of 11 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

SECTION-A

D'Moivre's theorem, application of D'Moivre's theorem including primitive nth root of unity. Expansions of $\sin n\theta$, $\cos n\theta$, $\sin^n\theta$, $\cos^n\theta$ $(n \in N)$. The exponential, logarithmic, direct and inverse circular and hyperbolic functions of a complex variable. Summation of series including Gregory Series.

SECTION-B

Hermitian and skew-hermitian matrices, linear dependence of row and column vectors, row rank, column rank and rank of a matrix and their equivalence. Theorems on consistency of a system of linear equations (both homogeneous and non-homogeneous). Eigen-values, eigen-vectors and characteristic equation of a matrix, Cayley-Hamilton theorem and its use in finding inverse of a matrix. Diagonalization.

Head,
Department of Mathematics
Punjabi University, Patiala

REFERENCES:

1. K.B. Datta : Matrix and Linear Algebra, Prentice Hall of India Pvt. Ltd., New Delhi, 2000.

2. S. R. Knight and H.S. Hall : Higher Algebra, H.M. Publications, 1994.

3. R.S. Verma and K.S. Shukla : Text Book on Trigonometry, Pothishala Pvt. Ltd., Allahabad.

4. Shanti Narayan and P.K. Mittal : A Text Book of Matrices, S. Chand & Co., New Delhi, Revised Edition, 2007.

Head,
Department of Mathematics
Punjabi University, Patiala

LESSON NO. 1.1 **CONVERTED INTO SLM MODE : DR. CHANCHAL**

DE-MOIVRE'S THEOREM AND ITS APPLICATIONS

Structure :

1.1.1 Objectives

1.1.2 Introduction to Basic Concepts (Complex Numbers)

1.1.2.1 The Product and Quotient of two Complex Numbers

1.1.3 De-Moivre's Theorem

1.1.4 The nth Roots of a Complex Number

1.1.5 nth Roots of Unity

1.1.6 Expressions for sin nθ and cosnθ in terms of Powers of cos θ and sin θ

1.1.7 Expressions for $\cos^n\theta$ and $\sin^n\theta$ in terms of multiples of cos θ and sin θ

1.1.8 Solution of Some Polynomial Equations

1.1.9 Summary

1.1.10 Key Concepts

1.1.11 Long Questions

1.1.12 Short Questions

1.1.13 Suggested Readings

1.1.1 Objectives

In this lesson, we will focus on algebra of complex numbers, De-Moivre's theorem and its applications, nth roots of unity.

1.1.2 Introduction to Basic Concepts (Complex Numbers)

A number of the form a + ib where a, b are reals and $i = \sqrt{-1}$, is called a complex number a is called the real part of Z = a + ib and b is called the imaginary part of Z.

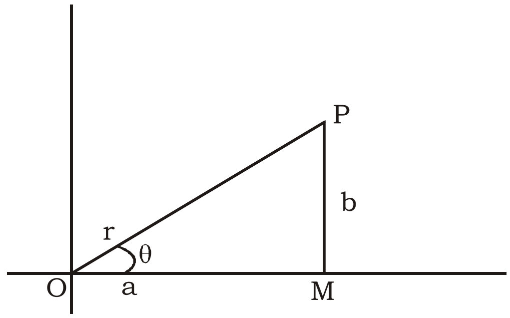
$\overline{Z}$ = a – ib is called the conjugate of Z.

If we take a = r cos θ and b = r sin θ, then r is called the modulus of Z and θ is called the argument of Z.

i.e. a = r cos θ, b = r sin θ,

$r = \sqrt{a^2 + b^2}$ and

$$\theta = \tan^{-1}\frac{b}{a}$$



1.1.2.1 The Product and Quotient of two Complex Numbers

Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

Now, $z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2]$

$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ .... (1)

and

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)}\frac{\left(\cos\theta_2 - i\sin\theta_2\right)}{\left(\cos\theta_2 - i\sin\theta_2\right)}$$

$$= \frac{r_1}{r_2}\left[\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) - i\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\right]$$

$$= \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) - i\sin\left(\theta_1 - \theta_2\right)\right] \qquad \text{............... (2)}$$

From the right hand sides of (1) and (2) we observe that, the absolute value of $z_1 z_2$ is $r_1 r_2$ and of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$ and amplitudes are $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ respectively. In other words absolute value of product (quotient) of two non zero complex numbers z_1 and z_2 is product (quotient) of their absolute values and the sum (difference) of every pair of values of the arguments of z_1, z_2 is an argument of $z_1 z_2 \left(\frac{z_1}{z_2}\right)$.

Symbolically, $|z_1 z_2| = |z_1||z_2|$, $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

and, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg } z_1 - \text{Arg } z_2$

$(z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)])$

For n variables : The result can be extended by induction.

If $z_1, z_2 \ldots\ldots\ldots\ldots z_n$ are non-zero complex numbers with polar forms,

i.e. $z_i = ri\,[(\cos\theta_i + i \sin\theta_i),\ 1 \le i \le n$

then,

$z_1 - z_2 \ldots\ldots\ldots\ldots z_n = r_1 r_2 \ldots\ldots\ldots r_n$

$([\cos(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n) + i \sin(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n)])$ ……………… (3)

when, in particular $z_1 = z_2 = \ldots\ldots\ldots\ldots\ldots = z_n = z,$

$r_1 = r_2 \ldots\ldots\ldots = r_n = r,\ \theta_1 = \theta_2 = \ldots\ldots\ldots\ldots\ldots = \theta_n = \theta$

then

$z^n = [r(\cos\theta + i \sin\theta)]^n$

$= r, r \ldots\ldots\ldots$ n times $[\cos(\theta + \theta \ldots\ldots\ldots\ldots$ n times$) + i \sin(\theta + \theta + \ldots\ldots\ldots$ n times$)]$

$=$ $z^n = r^n(\cos n\theta + i \sin n\theta)$ ……………… (4)

The relation (4) is true when n is a +ve integer.

If n is a negative integer, say n = –m, then

$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)} \cdot \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$= \cos m\theta - i \sin m\theta$

$= \cos(-m\theta) + i \sin(-m\theta)$

replacing – m by n

$= \cos n\theta + i \sin n\theta$

so we have proved the following theorem :

1.1.3 De-Moivre's theorem

If n is an integer (+ve or –ve)

then, $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

Note : Remember that $(\sin\theta + i \cos\theta)^n \neq \sin n\theta + i \cos n\theta$.

1.1.4 The nth Roots of a Complex Number

Suppose A is any complex number and we are interested to find the nth roots of A.

Let $Z = (A)^{\frac{1}{n}}$

then $Z^n = A$ .... (1)

Now Z is a complex number, say

$Z = r(\cos\theta + i\sin\theta)$

and $A = p(\cos\phi + i\sin\phi)$

Substituting the value of Z and A in (1), we get

$[r(\cos\theta + i\sin\theta)]^n = p(\cos\phi + i\sin\phi)$

By using the De-Moivre's theorem for +ve integers, we have

$r^n(\cos n\theta + i\sin n\theta) = p(\cos\phi + i\sin\phi)$

Now we know that, two complex numbers are equal if and only if their absolute values are same and their arguments differ by integral multiples of π.

So, $r^n = p$ and $n\theta = \phi + 2k\pi$, where k is any integer

Therefore roots are

$$p^{\frac{1}{n}}\left[\cos\left(\frac{\phi}{n} + \frac{2k}{n}\pi\right) + i\sin\left(\frac{\phi}{n} + \frac{2k}{n}\pi\right)\right] \qquad \text{............ (2)}$$

where k is any integer.

For k = 0, 1, 2, ... n – 1, the complex numbers (2) are all distinct since their arguments do not differ by integral multiples of 2π. Every other member occurring after the above n values of k will be equal to one of these.

For example, if we put $k = nq + m$, $0 \le m \le n$

$$\frac{\phi}{n} + \frac{2k}{n}\pi = \frac{\phi}{n} + \frac{2(nq+m)\pi}{n} = \frac{\phi}{n} + \frac{2m\pi}{n} + 2\pi q$$

Thus the n roots of A are given by

$$Z = p^{\frac{1}{n}}\left[\cos\left(\frac{\phi}{n} + \frac{2k}{n}\pi\right) + i\sin\left(\frac{\phi}{n} + \frac{2k}{n}\pi\right)\right]$$

where k = 0, 1, 2, ..., n – 1, and all are distinct.

Remark : From the above, we have seen that we can find the value of $\left[p(\cos\phi + i\sin\phi)\right]^{\frac{1}{n}}$ i.e. nth roots of a complex number can be determined.

For any rational number $\frac{m}{n}, n > 0$

$$Zm/n = Z^{1/n} = \left[n\left(\cos\phi + i\sin\phi\right)\right]^{\frac{1}{n}} = n\frac{m}{n}\left(\cos\phi + i\sin\phi\right)^{\frac{1}{n}}$$

then the n roots of $(\cos\phi + i\sin\phi)$ are

$$\cos\left(\frac{m\,\phi + 2k\pi}{n}\right) + i\sin\left(\frac{m\phi + 2k\pi}{n}\right) k = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots n-1$$

In particular when k = 0 we have

$$\left(\cos\theta + i\sin\theta\right)^{\frac{m}{n}} = \left(\cos\frac{m}{n}\theta + i\sin\frac{m}{n}\theta\right)$$

Hence we can say that De-Moivre's theorem also holds good when n is a rational number.

Example 1 : Simplify $\dfrac{(\cos\theta + i\sin\theta)^3\,(\cos\theta - i\sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^4}$

Sol. : $(\cos\theta - i\sin\theta)^5 = [\cos(-\theta) + i\sin(-\theta)]^5$

$= [(\cos\theta + i\sin\theta)^{-1}]^5$ (by De-Moiver's th^{m})

$= (\cos\theta + i\sin\theta)^{-5}$

$(\cos 2\theta + i\sin 2\theta)^4 = [(\cos\theta + i\sin\theta)^2]^4$

$= (\cos\theta + i\sin\theta)^8$

$$\therefore \quad \frac{(\cos\theta + i\sin\theta)^3\,(\cos\theta - i\sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^4}$$

$$= \quad \frac{(\cos\theta + i\sin\theta)^3\,(\cos\theta + i\sin\theta)^{-5}}{(\cos 2\theta + i\sin 2\theta)^8}$$

$= \quad (\cos\theta + i\sin\theta)^{-10} = (\cos 10\theta - i\sin 10\theta)$

Example 2 : Simplify $(\sin\theta + i\cos\theta)^6$

Solution : $\sin\theta + i\cos\theta = i\left(\cos\theta + \frac{1}{i}\sin\theta\right) = i(\cos\theta - i\sin\theta)$

$= i(\cos(-\theta) + i\sin(-\theta))$

$= i[\cos\theta + i\sin\theta]^{-1}$

$\therefore \quad (\sin\theta + i\cos\theta)^6 = [i(\cos\theta + i\sin\theta)^{-1}]^6$

$= i^6 (\cos\theta + i\sin\theta)^{-6}$

$= (-1)^3 (\cos 6\theta - i\sin 6\theta)$

$= -(\cos 6\theta - i\sin 6\theta) = -\cos 6\theta + i\sin 6\theta$

Alternative Method :

$$\sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$

$\Rightarrow$ $$[\sin\theta + i\cos\theta]^6 = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^6$$

Using De-Moivre's theorem

$$[\sin\theta + i\cos\theta]^6 = \cos\left(6\frac{\pi}{2} - 6\theta\right) + i\sin\left(6\frac{\pi}{2} - 6\theta\right)$$

$= \cos(3\pi - 6\theta) + i\sin(3\pi - 6\theta)$

$= -\cos 6\theta + i\sin 6\theta$

Example 3 : If $x + \frac{1}{x} = 2\cos\theta$, find $x^n + \frac{1}{x^n}$ and $x^n - \frac{1}{x^n}$ where n is a positive integer.

Solution : We have $x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow$ $x^2 + 1 = 2x\cos\theta$

$\Rightarrow$ $x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow$ $$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2i\sin\theta}{2}$$

$= \cos\theta \pm i\sin\theta$

when we take $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = (\cos\theta + i\sin\theta)^{-1}$$

$\therefore \quad x^n + \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin \theta$

$= 2 \cos n\theta$

and

$$x^n - \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$$

$= (\cos n\theta + i \sin n\theta) - (\cos n\theta + i \sin n\theta)$

$= 2 i \sin n\theta$

Hence

$$x^n + \frac{1}{x^n} = 2\cos n\theta$$

and

$$x^n - \frac{1}{x^n} = 2i \sin n\theta$$

sly, we can find the values for $x = \cos \theta - i \sin \theta$

Example 4 : Show that if $(\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) = 1$ then $\alpha + \beta = 2k \pi$ where k is any integer.

Solution : Now $(\cos \alpha + i \sin \alpha) (\cos \beta + i \sin \beta) = 1$

or $\cos (\alpha + \beta) + i \sin (\alpha + \beta) = 1$

or $\cos (\alpha + \beta) = 1$ and $\sin (\alpha + \beta) = 0$

[Equating real and imaginary parts]

Hence $\alpha + \beta = 2k \pi$ where k is an integer.

Example 5 : Prove that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$

$$\text{L.H.S.} = \left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)}\right]^n$$

$$= \left[\frac{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + 2i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + 2i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} \right]^n$$

$$= \left[\frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} \right]^n$$

$$\frac{\left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]^n}{\left[\cos\left(-\frac{\pi}{4} + \frac{\theta}{2}\right) + i\sin\left(-\frac{\pi}{4} + \frac{\theta}{2}\right)\right]^n}$$

$$= \left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]^n . \left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]^n$$

$$= \left[\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]^{2n}$$

$$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right) = \text{R.H.S}$$

Example 6 : Apply De-Moivre's theorem to find the quadratic whose roots are nth powers of theroots of $x^2 - 2x\cos\theta + 1 = 0$

Solution : $x^2 - 2x\cos\theta + 1 = 0$, $x = \cos\theta \pm i\sin\theta$

If the take $\alpha = \cos\theta + i\sin\theta$ and $\beta = \cos\theta - i\sin\theta$

then, $\alpha^n = \cos n\theta + i\sin n\theta$ and $\beta^n = \cos n\theta - i\sin n\theta$

S = sum of roots = $\alpha^n + \beta^n = 2\cos n\theta$

P = product of roots = $\alpha^n\beta^n = 1$

The required quadratic is

$x^2 - Sx + P = 0$

$x^2 - 2x \cos n\theta + 1 = 0$

Example 7 : If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$

Then show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(iii) $\sum \sin^2\alpha = \sum \cos^2\alpha = \frac{+3}{2}$

Solution : Given $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$

We define $\alpha = \text{cis}\,\alpha = \cos\alpha + i\sin\alpha$

$b = \text{cis}\,\beta, \qquad c = \text{cis}\,\gamma$

$\therefore \quad a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$a + b + c = 0$

We also know that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

i.e. $(\text{cis}\,\alpha)^3 + (\text{cis}\,\beta)^3 + (\text{cis}\,\gamma)^3 = 3\,\text{cis}\,\alpha\,\text{cis}\,\beta\,\text{cis}\,\gamma$

$\text{cis}\,3\alpha + \text{cis}\,3\beta + \text{cis}\,3\gamma = 3\,\text{cis}(\alpha + \beta + \gamma)$

On comparing real and imaginary parts, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$ and

$\sin 3\alpha + \sin 3\beta + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

To prove last part :

since $a + b + c = 0$ and $a = \text{cis}\,\alpha$, $b = \text{cis}\,\beta$, $c = \text{cis}\,\gamma$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \text{cis}\left(-\alpha + \text{cis}(-\beta) + \text{cis}(-\gamma)\right)$$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = \theta$

$\Rightarrow \quad ab + bc + ca = 0$

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 = 0 \qquad [\because ab + bc + ca = 0]$

$\Rightarrow \quad \text{cis}\,2\alpha + \text{cis}\,2\beta + \text{cis}\,2\gamma = 0$

i.e. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$

$$\Rightarrow \quad 2\left(\cos^2 + \cos^2\beta + \cos^2\gamma\right) = \frac{3}{2}$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

Similarly we can prove that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$$

Example 8 : a = cis α, b = cis β, c = cis γ then prove that

(i) $\frac{ab}{c} + \frac{c}{ab} = 2\cos(\alpha+\beta-\gamma)$

(ii) $a^p b^q c^r + \frac{1}{a^p b^q c^r} = 2\cos(\alpha p + \beta q - \gamma r)$

(iii) $a^p b^q c^r + \frac{1}{a^p b^q c^r} = 2\,i\sin(\alpha p + \beta q - \gamma r)$

Solution :

(i) $\frac{ab}{c} + \frac{c}{ab} = \frac{\text{cis}\,\alpha\ \text{cis}\,\beta}{\text{cis}\,\gamma} + \frac{\text{cis}\,\gamma}{\text{cis}\,\alpha\ \text{cis}\beta}$ $[\because a = \text{cis}\,\alpha, b = \text{cis}\,\beta, c = \text{cis}\,\gamma]$

=cos ($\alpha + \beta - \gamma$) + cis (γ) – (α) – (β)

= cos ($\alpha + \beta - \gamma$) + i sin ($\alpha + \beta - \gamma$) + cos ($\alpha + \beta - \gamma$) – i sin ($\alpha + \beta - \gamma$)

$$\left[\because \begin{array}{l}\cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta\end{array}\right]$$

= 2 cos ($\alpha + \beta - \gamma$)

In a similar manner (ii) and (iii) parts can be proved.

Example 9 : Prove that $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}$ where n is any positive integer.

Solution : $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\frac{\cos\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore \quad (1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$$= 2^{\frac{n}{2}}\left[\sqrt{2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right]$$

also

$$1 = i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$\therefore \qquad (1-i)^n = \left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)\right]^n$$

$$= 2^{\frac{n}{2}}\left[\left(\cos\left(\frac{-n\pi}{4}\right) + i\sin\left(\frac{-n\pi}{4}\right)\right)\right]$$

Now

$$\cos\left(\frac{-n\pi}{4}\right) = \cos\frac{n\pi}{4} \text{ and } \sin\left(\frac{-n\pi}{4}\right) = -\sin\frac{n\pi}{4}$$

Hence

$$(1 + i)^n + (1-1)^n$$

$$= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4} = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$$

Example 10 : Find the two square roots of $\left(\frac{1+i}{\sqrt{2}}\right)$

Solution : $\left[\frac{1+i}{\sqrt{2}}\right]^{\frac{1}{2}} = \left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]^{\frac{1}{2}}$

$$= \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{\frac{1}{2}} = \cos\frac{\frac{\pi}{4} + 2k\pi}{2} + i\sin\frac{\frac{\pi}{4} + 2k\pi}{2}$$

where k = 0, 1

where k = 0, one root is $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ $\left[\frac{\pi}{8} = \frac{180}{8} = 22.50\right]$

when k = 1, the other root is $\cos\left[\frac{\frac{\pi}{4}+2\pi}{2}\right] + i\sin\left[\frac{\left(\frac{\pi}{4}+2\pi\right)}{2}\right]$

$$= \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$$

Now $\cos\left(\frac{9\pi}{8}\right) = \cos\left(\pi + \frac{\pi}{8}\right) = -\frac{\cos\pi}{8}$

and $\sin\frac{9\pi}{8} = \sin\left(\pi + \frac{\pi}{8}\right) = -\sin\frac{\pi}{8}$

Hence two square roots are

$$\pm\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Self Check Exercise

1. Prove that : $(1+i)^n - (1-i)^n = \frac{2}{2^{\frac{n+2}{2}}}\sin\frac{n\pi}{8}$

....
....

1.1.5 nth Roots of Unity

When $n \geq 1$ and n is an integer, then any complex number Z such that $Z^n = 1$ is called on nth root of unity.

Now

$$[p(\cos\theta) + i\sin\theta]^{\frac{1}{n}} = p^{\frac{1}{n}}\left[\cos^{\frac{\theta+2k\pi}{n}} + i\sin^{\frac{\theta+2k\pi}{n}}\right]$$

where k = 0, 1, 2, ..., n – 1

In this case, p (cos θ + i sin θ) = 1

i.e. p = 1 and θ = 0

Hence nth roots of unity are $\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, where k = 0, 1, 2 ...., n –1

if $a=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$

then, $1 = a^0, a^1, a^2 \ldots\ldots, a^{n-1}$ (since $a^n = 1$)

are the nth roots of unity.

Example 11 : Find the sixth roots of unity, which of these are square roots of unity ?

Solution : Roots are given by $\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$ where k = 0, 1, 2, 3, 4, 5.

when k = 0, $\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}=1$

when k = 1, $\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}=\frac{1+i\sqrt{3}}{2}$

when k = 2, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)+i\sin\left(\pi-\frac{\pi}{3}\right)$

$$=-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

when k = 3, cos π + i sin π = –1

when k = 4, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=\cos\left(\pi+\frac{\pi}{3}\right)+i\sin\left(\pi+\frac{\pi}{3}\right)=-\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$

when k = 5,

$$\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\cos\left(2\pi+\frac{\pi}{3}\right)+i\sin\left(2\pi+\frac{\pi}{3}\right)=-\cos\frac{\pi}{3}=i\sin\frac{\pi}{3}=\frac{-1}{2}-i\frac{\sqrt{3}}{2}=-\frac{\left(1+i\sqrt{3}\right)}{2}$$

Out of these six roots, 1, –1 are square roots of 1.

1.1.6 Expressions for sinnθ and cosnθ in terms of Powers of cos θ and sin θ : n being a positive integer.

We already know that

$\sin 2\theta = 2 \sin\theta \cos\theta$ (formula)

$\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$ (formula)

$\cos 2\theta = 2 \cos^2\theta - 1$ (formula)

$\cos 3\theta = 4 \cos^2\theta - 3 \cos\theta$ (formula)

Thus we have been able to express $\sin 2\theta$, $\sin 3\theta$, $\cos 2\theta$, $\cos 3\theta$ in the powers of $\sin\theta$ and $\cos\theta$.

Now we find expansions of $\sin n\theta$ and $\cos n\theta$ in terms of $\sin\theta$ and $\cos\theta$.

By De-Moivre's theorem

$$\cos n\theta + i \sin\theta = (\cos\theta + i \sin\theta)^n$$

Now expanding $(\cos\theta + i\sin\theta)^n$ by binomial theorem, we have

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + {}^nc_1 \cos^{n-1}\theta (i\sin\theta)$$
$$+ {}^nc_2 \cos^{n-2}\theta (i\sin\theta)^2 + {}^nc_3 (\cos\theta)^{n=r} (i\sin\theta)^3$$
$$+ \ldots\ldots\ldots\ldots + {}^nc_r (\cos\theta)^n - r\,(i\sin\theta)^r + \ldots\ldots\ldots\ldots {}^nc_n (i\sin\theta)^n$$

or

$$(\cos n\theta + i\sin n\theta) = [\cos^n\theta - {}^nc_2 \cos^{n-2}\theta \sin^2\theta + {}^nc_4 \cos^{n-4}\theta \sin^4\theta + \ldots\ldots\ldots\ldots]$$
$$+ i\,[{}^nc_1 \cos^{n-1}\theta \sin\theta - {}^nc_3 \cos^{n-3}\theta \sin^3\theta + {}^nc_5 \cos^{n-5}\theta \sin^5\theta + \ldots\ldots\ldots\ldots]$$

on equating real and imaginary parts we get the value of $\cos n\theta$ and $\sin n\theta$.

Example 12 : Express $\cos 4\theta$ in terms of powers of $\cos\theta$

Solution : $\cos 4\theta = \cos^4\theta - {}^4c_2 \cos^2\theta \sin^2\theta + {}^4c_4 \sin^4\theta$

$$= \cos^4\theta - \frac{4!}{2!2!}\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$

$$= \cos^4\theta - 6\,(\cos^2\theta - \cos^4\theta) + (1 + \cos^4\theta - 2\cos^2\theta)$$

$$= 8\cos^4\theta - 6\cos^2\theta + 1.$$

Example 13 : Express $\sin 7\theta$ in terms of powers of $\sin\theta$

Solution : $(\cos\theta + i\sin\theta)^7 = \cos^7\theta + i^7 c_3 \cos^4\theta \sin^3\theta - {}^7c_2 \cos^5\theta \sin^2\theta - i^7 c_3 \cos^4\theta \sin^3\theta$

$+ {}^7c_4 \cos^3\theta \sin^4\theta + i.\ {}^7c_5 \cos^2\theta \sin^5\theta - {}^7c_6 \cos\theta \sin^6\theta + i\ {}^7c_7 \sin^7$

an equating imaginary party, we get

$$\sin 7\theta = {}^7c_1 \cos^6\theta \sin\theta - {}^7c_3 \cos^4\theta \sin^3\theta + {}^7c_5 \cos^2\theta \sin^5\theta - {}^7c_7 \sin^7\theta$$

$$= 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin\theta$$

$$= 7\,(1-\sin^2\theta)^3 .\ \sin\theta - 35\,(1-\sin^2\theta)^2 \sin^3\theta + 2I\,(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$$

$$= 7\,(1 - \sin^6\theta - 3\sin^2\theta + 3\sin^4\theta)\sin\theta$$
$$- 35\,(1 + \sin^4\theta - 2\sin^2\theta \sin^3\theta)$$
$$+ 21\,(\sin^5\theta - \sin^7\theta) - \sin^7\theta$$

$$= 7\,[\sin\theta - \sin^7\theta - 3\sin^3\theta + 3\sin\theta]$$
$$-35\,[\sin^3\theta + \sin^7\theta - 2\sin^5\theta]$$
$$+ 21\,[\sin^5 a - \sin^7\theta] - \sin^7\theta$$

$= -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$

1.1.7 Expressions for $\cos^n\theta$ and $\sin^n\theta$ in terms of multiples of $\cos\theta$ and $\sin\theta$

If $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$

by De-Moivre's theorem

$z^n = \cos n\theta + i\sin n\theta \ \frac{1}{z^n} = \cos n\theta - i\sin n\theta$

so that

$$z + \frac{1}{z} = 2\cos\theta,\ z - \frac{1}{z} = 2i\sin\theta$$

$$z + \frac{1}{z^n} = 2\cos\theta,\ z - \frac{1}{z^n} = 2i\sin\theta$$

Now,

$$(2\cos\theta)^n = \left(z + \frac{1}{z}\right)^n$$

$$= z^n + {}^nC_1 z^n \cdot \left(\frac{1}{z}\right) + {}^nC_2\, z^n \left(\frac{1}{z}\right)^2$$

$$+ \dots\dots\dots\dots + {}^nC_r\, z^n \left(\frac{1}{z}\right)^r + \dots\dots\dots\dots + {}^nC_n \left(\frac{1}{z}\right)^n$$

$$\left(z^n + \frac{1}{z^n}\right) + {}^nC_1\left(z^{n-1} + \frac{1}{z^{n-1}}\right) + {}^nC_2\left(z^{n-2} + \frac{1}{z^{n-2}}\right) + \dots\dots\dots$$

$\Rightarrow \quad 2^n\cos^n\theta = 2\cos n\theta + {}^nC_1\, 2\cos(n-1)\theta + {}^nC_2\, 2\cos(n-2)\theta + \quad \dots\dots\dots (1)$

OR

$2^{n-1}\cos^n\theta = \cos n\theta + {}^nC_1\cos(n-1)\theta + {}^nC_2\cos(n-2)\theta + \quad \dots\dots\dots\dots (2)$

Similarly, the expression can be derived for $\sin^n\theta$.

Example 14 : Express $\cos^6\theta$ in terms of those of cosines of multiples of θ.

Solution : Let $z = \cos\theta + i\sin\theta$

than, $\frac{1}{z} = \cos\theta - i\sin\theta$

On adding, $2\cos\theta = z + \frac{1}{z}$

$$\Rightarrow \qquad 2^6 \cos^6 \theta = \left(z + \frac{1}{z}\right)^6$$

$$= z^6 + {}^6C_1 z^4 + {}^6C_2 z^2 + {}^6C_3 + {}^6C_4 z^{-2} + {}^6C_5 z^{-4} + {}^6C_6 z^{-6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + {}^6C_1 \left(z^4 + \frac{1}{z^4}\right) + {}^6C_2 \left(z^2 + \frac{1}{z^2}\right) + {}^6C_3$$

$$= 2 \cos 6\theta + 6.2 \cos 4\theta + 15.2 \cos 2\theta + 20$$

$$2^6 \cos^6 \theta \, [2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20]$$

$$\cos^6 = \frac{1}{32}\left[\cos 6\theta + 5\cos 4\theta + 15\cos 2\theta + 10\right]$$

1.1.8 Solution of Some Polynomial Equations

Example 15 : Find the solutions of $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

Solution : $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = \dfrac{x^8 - 1}{x - 1}$

$\therefore$ the solutions of the given equation are the roots of $x^8 = 1$ except $x = 1$.

Now $x^8 = 1$

$= 1. (\cos 2r\pi + i \sin 2r\pi)$

$$\Rightarrow \qquad x = \cos \frac{2r\pi}{8} + i \sin \frac{2r\pi}{8}, \text{ where } r = 0, 1, 2, \ldots\ldots\ldots\ldots, 7 .$$

when $r = 0$, $x = 1$

$$x = \cos \frac{2r\pi}{8} + i \sin \frac{2r\pi}{8}, \text{ where } r = 0, 1, 2, \ldots\ldots\ldots\ldots, 7$$

when $r = 0$, $x = 1$

$$r = 1, x = \cos \frac{2\pi}{8} + i\, Sin \frac{2\pi}{8} \ldots\ldots\ldots \text{ so on}$$

Hence the solutions of the given equation are given by

$$x = \cos \frac{2r\pi}{8} + i \sin \frac{2r\pi}{8}, \text{ where } r = 1, 2, \ldots\ldots\ldots\ldots, 7$$

Example 16 : Using the equation $x^6 + x^5 + \ldots\ldots\ldots\ldots + x + 1 = 0$, Find an equation whose roots are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}$.

Solution : As done in the previous example, we can find that

$\cos\frac{2r\pi}{7} + i\sin\frac{2r\pi}{7}$ are the roots of $x^6 + x^5 + \ldots\ldots\ldots\ldots\ldots x + 1 = 0$,

where $r = 0, 1, 2, \ldots\ldots\ldots 6$

Now, $\cos\frac{2\pi}{7} = \frac{e^{\frac{2\pi i}{7}} + e^{\frac{-2\pi i}{7}}}{2}, \cos\frac{4\pi}{7} = \frac{e^{\frac{4\pi i}{7}} + e^{\frac{-4\pi i}{7}}}{2}$

and $\cos\frac{6\pi}{7} = \frac{e^{\frac{6\pi i}{7}} + e^{\frac{-6\pi i}{7}}}{2}$

Let, $2y = x + \frac{1}{x}$

Then, $4y^2 = x^2 + \frac{1}{x^2} + 2$

$\Rightarrow$ $x^2 + \frac{1}{x^2} = 4y^2 - 2$

and $8y^3 = x^3 + \frac{1}{x^3} + 6y$

Now, if $x = e^{\frac{2\pi in}{7}}$, then $y = \cos\frac{2\pi n}{7}, n = 1, 2$

Also then

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

$$x^3 + x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} = 0$$

$$\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0$$

$8 - 6y + 4 - 2 + y + 1 = 0$

$8y^3 + 4y^2 - 5y - 1 = 0$

Hence $\cos \frac{2\pi n}{7}$, n = 1, 2, 3 are the root of

$8y^3 + 4y^2 - 5y - 1 = 0.$

1.1.9 Summary

In this lesson, we have discussed about the complex numbers and their important characteristics such as operations on complex numbers, nth roots of complex numbers. The concept is used to find out nth roots of unity and to solve some polynomial equations.

1.1.10 Key Concepts

Complex Numbers, De-Moivre's theorem, nth roots of complex numbers, nth roots of unity.

1.1.11 Long Questions

1. (a) Is $(\sin \theta + i \cos \theta)^n = \sin n\theta + i \cos n\theta$, n being a + ve integer ?

   (b) Is $(\sin \theta_1 + i \cos \theta_1)(\sin \theta_2 + i \cos \theta_2) = \sin(\theta_1 + \theta_2) + i \cos(\theta_1 + \theta_2)$? Justify

2. Prove that

   $$2\cos\frac{2\pi}{5}, 2\cos\frac{4\pi}{5}, 2\cos\frac{6\pi}{5}$$

   $2\cos\frac{8\pi}{5}$ are the roots of the equation

   $x^4 + 2x^3 - x^2 - 2x + 1 = 0$

3. Prove that the roots of

   $(1 + x)^{2n} + (1 - x)^{2n} = 0$ are given by

   $$x = \pm i \tan\frac{2\,im}{4\pi}\pi;\ m = 0, 1, 2, \ldots\ldots..(n-1)$$

4. Prove, that

   $\sin\frac{\pi}{14}$ is a root of

$64x^6 - 80x^4 + 24x^2 - 1 = 0$

5. Show that the roots of the equation

$(x-1)^n = x^n$ are $\frac{1}{2}\left(1 + i\cot\frac{\pi r}{n}\right)$

$r = 0, 1, 2, \ldots\ldots\ldots, (n-1)$

1.1.12 Short Questions

1. Simplify the following :

(a) $\dfrac{(\cos 4\theta + i\sin 4\theta)^3}{(\cos 2\theta - i\sin 2\theta)^2}$

(b) $(\cos 2\theta + i\sin 2\theta)^4$

(c) $(1 - \cos\theta + i\sin\theta)^8$

2. Find the four fourth roots of $\sqrt{3} + i$

3. Find the three cube roots of $-\sqrt{2} + \sqrt{2}\,i$

4. Find the eight eighth roots of 1.

5. Express the following in terms of $\cos\theta$ and $\sin\theta$
   1. $\sin 6\theta$
   2. $\cos 5\theta$
   3. $\sin 9\theta$

6. Express the following in terms of cosines or sines of multiples angles
   (a) $\sin^6\theta\cos^2\theta$
   (b) $\cos^7\theta\sin^2\theta$
   (c) $\sin^8\theta\cos^4\theta$

7. Solve the equation
   (a) $x^7 + 1 = 0$
   (b) $x^4 - x^3 + x^2 - x + 1 = 0$
   (c) $(1 + x)^5 + x^5 = 0$.

1.1.13 Suggested Readings

R.S. Verma and K.S. Shukla, Text book on Trigonometry, Pothishala Pvt. Ltd., Allahabad

---

LESSON NO. 1.2 **CONVERTED INTO SLM MODE : DR. CHANCHAL**

FUNCTIONS OF COMPLEX VARIABLE

1.2.1 Objectives
1.2.2 Introduction to the Series of Exponential Function
1.2.3 Periodicity of Exponential Function
1.2.4 Logarithmic Function
1.2.5 Laws of Logarithm
1.2.6 Gregory Series
1.2.7 Summary
1.2.8 Key Concepts
1.2.9 Long Questions
1.2.10 Short Questions
1.2.11 Suggested Readings

1.2.1 Objectives

In this lesson, we will focus on :

1. The concept of exponential functions related to complex variables and their periodicity.
2. Principal and general value of logarithmic functions.

1.2.2 Introduction to the Series of Exponential Function

Consider e^z $= e^{x+iy} = e^x e^{iy}$ where $e^{iy} = (\cos y + i \sin y)$

$= e^x (\cos y + i \sin y)$

Real (Re) part of $e^z = e^x \cos y$

Imagi e^{z1+z2} g) part of $e^z = e^x \sin y$

and the series for e^z is

$$e^z = I + z + \frac{z^2}{\lfloor 2_1} + \frac{z^3}{\lfloor 3} + \ldots\ldots\ldots\ldots$$

e^{z1+z2}

$$= 1 + (x+iy) + \frac{(x+iy)^2}{\lfloor 2} + \frac{(x+iy)^3}{\lfloor 3} + \ldots\ldots\ldots\ldots$$

Also $e^{z_1}e^{z_2} = e^{z_1} + e^{z_2}$

Take $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

$z_1 + z_2 = ((x_1 + x_{2)}) + i\ (y_1 + y_2))$

$$e^{z_1} + e^{z_2} = e^{x_2 + x_2}\left[\cos\left(y_1 + y_2\right) + i\sin\left(y_1 + y_2\right)\right]$$

$$= e^{x_1} e^{x_2}\left(\cos y_1 + i\sin y_1\right)\left(\cos y_2 + i\sin y_2\right)$$

$$= e^{x_1}\ .\left(\cos y_1 + i\ \sin y_1\right), e^{x_2}\left(\cos y_2 + i\ \sin y_2\right)$$

$$= e^{z_1}\ .\ e^{z_2}$$

Hence $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$

1.2.3 Periodicity of Exponential Function

We have seen above that

$e^z = e^x (\cos y + i \sin y)$

Since cos y & sin y are periodic functions

So, $e^z = e^x [\cos (2\pi r + y) + i \sin (2\pi r + y)]$

$= e^x .\ e^{(2r\pi + y)i}$

$= e^{x + iy + 2r\pi i}$

$= e^{z + 2r\pi i}$

Hence e^z is perodic with period $2r\pi$.

Example 1 : Find the imaginary part of $e^{(2-3i)^2}$

Sol. : $e^{(2-3i)^2} = e^{4+9i^2-12i}$

$= e^{-5-12i}$

$= e^{-5} (\cos 12 - i \sin 12)$

and hence, $\text{img.}\left(e^{(2-3i)^2}\right) = -e^{-5}\sin 12$. nb

Example 2 : Express $e^i + e^{-i}$ in the form A + iB.

Sol. : $e^i + e^{-i} = (\cos 1 + i \sin 1) + (\cos 1 - i \sin 1)$

$= 2 \cos 1$ $+ i$

So $e^i + e^{-i}$ is purely real.

Example 3 : If a and b are roots of $x^2 + x + 1 = 0$

Show that for any number n, $ae^{na} + be^{n^b} = e^{-\frac{n}{2}}\left(\cos\sqrt{\frac{3}{2}}n + \sqrt{3}\sin\frac{\sqrt{3}}{2}n\right)$

Sol. : Now clearly $\quad -a = \dfrac{-1i\sqrt{3}}{2}$

$$b = \frac{-\left(1 + i\sqrt{3}\right)}{2} \qquad n$$

Now $\quad ae^{na} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{n\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)} \quad n$

$$= \left(\frac{-1}{2} + \frac{i}{2}\sqrt{3}\right) e^{-\frac{n}{2}} \left(\cos\frac{\sqrt{3}n}{2} + i\sin\frac{\sqrt{3}}{2}\right)$$

$$= e^{-\frac{n}{2}} \left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right] \left(\cos\frac{\sqrt{3}}{2} + i\sin\frac{\sqrt{3}}{2}\right)$$

Similarly, $\quad be^{nb}\ e^{-\frac{n}{2}} \left[\frac{-1}{2} = i\frac{\sqrt{3}}{2}\right] \left[\cos\frac{\sqrt{3}n}{2} - \sin\frac{\sqrt{3}n}{2}\right]$

On adding the last two, we get the desired result.

1.2.4 Logarithmic Function : (Principal Value)

$z = x + iy = re^{i\theta}$ where $r = \sqrt{x^2 + y^2}\ \left(\text{or } r^2 = x^2 + y^2\right)$

and $\quad \theta = \tan^{-1}\dfrac{y}{x}$

Now, $\log z = \log(x + iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\log x^2 + y^2 + i\tan^{-1}\frac{y}{x}$$

1.2.5 Laws of Logarithm

General value of $\log z$ is given by adding $2n\pi i$ to principal value of $\log z$

i.e. $\log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} + 2n\pi i$

Example 4 : Find the general and principal value of log i.

Sol. : log i = log (0 + 1 i)

$$= \frac{1}{2}\log(0^2+1^2) + i\tan^{-1}\left(\frac{1}{0}\right)$$

$$= \frac{1}{2}\log 1 + i\tan^{-1}\left(\frac{1}{0}\right)$$

$$= 0 + i\frac{\pi}{2}$$

$\Rightarrow$ $\log i = i\frac{\pi}{2}$ (Principal value)

Now, its general value is given by $-\frac{\pi}{2} + 2n\pi i$

$$= (4n+1)\frac{\pi}{2}i$$

Example 5 : Express log (log i) in the form of A + iB

Solution : $\log i = (4n+1)\frac{\pi}{2}i$

Also, log [log i] = log $(4n+i\ )\frac{\pi}{2}i + \frac{\pi}{2}i + 2m\pi i$

$$= \log(4n+1)\frac{\pi}{2} + i(4m+1)\frac{\pi}{2}$$

Example 6 : Prove that $\log(1+r\,\mathrm{cis}\theta) = \frac{1}{2}\log(1+2r\cos\theta+r^2\cos^2\theta) + i\tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$

Solution : log (1 + r cisθ) = log 1 + r cos θ + ir sin θ)

$$= \frac{1}{2}\log\left[(1+r\cos\theta)^2 + (r\sin\theta)^2\right] + i\tan^{-1}\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$$

$$= \frac{1}{2}\log\left(1 + r^2\cos\theta + 2r\cos\theta\right) + \tan^1\left(\frac{r\sin\theta}{1 + r\cos\theta}\right)$$

$$= \frac{1}{2}\log\left(1 + 2r\cos\theta\right) + i\tan^{-1}\left(\frac{r\sin\theta}{1 + r\cos\theta}\right)$$

1.2.6 Gregory's Series

If $-1 \le x \le 1$, then

$$\tan^{-1} x = x - \frac{x^2}{3} + \frac{x^5}{5}............\infty$$

or when $\frac{-\pi}{4} \le \theta \le \frac{\pi}{4}$, then

$$\theta = (\tan\theta) - \frac{1}{3}(\tan\theta)^3 + \frac{1}{5}(\tan\theta)^5 ....\infty$$

Proof : consider

$$\log(1 + ix) = \frac{1}{2}\log(1 + x^2) + i\tan^{-1}(x) \qquad .... (1)$$

Also $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3}............$

and $\log(1 + ix) = ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3}................$

$$= ix + \frac{x^2}{2} + i\frac{x^3}{3}........... (2)$$

on comparing (1) and (2), we get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5}..............$$

if we put $\tan^{-1} x = \theta$ or $x = \tan\theta$ we get

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta..........$$

if we put $\theta = \frac{\pi}{4}$, then

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5}$......... (an important result to remember)

Application of Gregory series to find the value of π

consider $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}-\frac{1}{3}}\right) = \tan^{-1} 1 = \frac{\pi}{4}$

so, $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$$= \left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 \ldots\ldots\ldots\ldots\right) + \left(\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^{\frac{1}{3}} + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \ldots\ldots\ldots\ldots\ldots\right)$$

= 0.7854 (on simplification)

Example 7 : Prove that $\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\ldots\ldots\ldots\right]$

Solution : First find the value of R.H.S. and then compare

RHS = $2\sqrt{3}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\ldots\ldots\ldots\right]$

$$= 2\sqrt{3}\left[1 - \frac{1}{3\left(\sqrt{3}\right)^2} + \frac{1}{5}\,\frac{1}{\left(\sqrt{3}\right)^4} - \frac{1}{7}\cdot\frac{1}{\left(\sqrt{3}\right)^6}\ldots\ldots\ldots\ldots\right]$$

$$= 2 \times 3\left[\frac{1}{\sqrt{3}} - \frac{1}{3}\cdot\frac{1}{\left(\sqrt{3}\right)^3} - \frac{1}{5}\cdot\frac{1}{\left(\sqrt{3}\right)^5}\dots\dots\dots\right]$$

$$= 6\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 6 \times \frac{\pi}{6}$$

$= \pi =$ L.H.S.

Example 8 : Prove that $\left(\frac{2}{3}+\frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)\dots\dots\dots = \frac{\pi}{4}$

Solution : $\left(\frac{2}{3}+\frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)\dots\dots\dots = \frac{\pi}{4}$

L.H.S. $= 2\left[\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5}\dots\dots\dots\dots\right] + \left[\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{5}\cdot\frac{1}{7^5}\dots\dots\dots\right]$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{7}$$

$$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$\tan^{-1}(1) = \frac{\pi}{4}$ R.H.S.

Self Check Exercise

1. Use Gregory series to show that

$$\left(1-\frac{1}{3^{\frac{1}{2}}}\right)-\frac{1}{3}\left(1-\frac{1}{3^{\frac{3}{2}}}\right)+\frac{1}{5}\left(1-\frac{1}{3^{\frac{5}{2}}}\right)+\ldots\ldots\ldots\ldots\ldots=\frac{\pi}{12}$$

....
....
....

1.2.7 Summary

From this lesson, we have gained the knowledge of exponential functions of complex variables which will help us to understand the concept of hyperbolic functions of complex variables and to find the sum of finite trigonometric series. Further, based on the exponential functions, we have defined the logarithmic functions of complex variables.

1.2.8 Key Concepts

Exponential functions of complex variables, logarithmic functions of complex variables, Gregory's series.

1.2.9 Long Questions

1. If $-\left(\sqrt{2}-1\right) < x < \sqrt{2}-1$, show that

$$2\left(x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\ldots\ldots\right)=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5$$

2. If x > 0, prove that

$$\tan^{-1}x=\frac{\pi}{4}+\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+5}\right)^5$$

1.2.10 Short Questions

1. Split into real and imaginary parts

(a) e^{z^2} (b) $e^{(6-5i)^2}$

2. Define periodic functions and give an example.

1.2.11 Suggested Readings

R.S. Verma and K.S. Shukla, Text book on Trigonometry, Pothishala Pvt. Ltd., Allahabad

LESSON NO. 1.3 **CONVERTED INTO SLM MODE : DR. CHANCHAL**

SUMMATION OF SERIES AND HYPERBOLIC FUNCTIONS

1.3.1 Objectives
1.3.2 Summation of Finite Trigonometric Series
1.3.3 Some Important EXamples
1.3.4 C + iS Method
1.3.5 Hyperbolic Functions of a Complex Variable
1.3.6 Summary
1.3.7 Key Concepts
1.3.8 Long Questions
1.3.9 Short Questions
1.3.10 Suggested Readings

1.3.1 Objectives

In this lesson, we will learn to evaluate the sum of finite trigonometric series. Further, we will study direct and inverse circular and hyperbolic functions of a complex variable.

1.3.2 Summation of Finite Trigonometric Series

Art : Find the sum of cos α + cos (α + β) + cos (+ 2β) + ...

$$................+\cos\left(a+\overline{n+1}\,\beta\right)$$

Sol. : Let Denote it by 'S' as

$$S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... +\cos\left(a+\overline{n+1}\,\beta\right)$$

Multiplying both sides by sin $\frac{\beta}{2}$

$$S\,.\,\sin\frac{\beta}{2} = \cos\alpha\,\sin\frac{\beta}{2} + \cos(\alpha+\beta)\sin\frac{\beta}{2} + \cos(\alpha+2\beta)\frac{\sin\beta}{2} + ...$$

$$.............+\cos\left(\alpha+\overline{n+1}\,\beta\right)\sin\frac{\beta}{2}$$

Using the result $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

We have

$$S.\sin\frac{\beta}{2} = \frac{1}{2}\left[\left\{\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)\right\}+\left\{\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)\right\}\right]+\ldots\ldots\ldots$$

$$+\left\{\sin\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)-\sin\left(\alpha+\frac{n-3}{2}\beta\right)\right\}$$

$$=\frac{1}{2}\left[\sin\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)-\sin\left(\alpha+\frac{\beta}{2}\right)\right]$$

$$=\frac{1}{2}\left[2\cos\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)\sin\frac{n\beta}{2}\right]$$

$$S=\frac{\cos\left(\alpha+\frac{\overline{n-1}}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\alpha}{2}}\ldots\ldots\ldots\ldots\text{(A)}$$

Cor. When $\beta = \alpha$

$$\cos\alpha+\cos 2\alpha+\ldots\ldots\ldots\ldots+\cos n\alpha$$

$$=\frac{\cos\left(\alpha+n-1\,.\,\frac{\alpha}{2}\right)\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$=\frac{\cos\left(x-n+1\;\frac{\alpha}{2}\right)\sin n\frac{x}{2}}{\sin\frac{x}{2}}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\text{(A')}$$

Similarly we can prove that

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots\ldots\ldots + \sin\left(\alpha + \overline{n-1}\,\beta\right),$$

$$= \frac{\cos\left[\alpha + \left(\frac{n+1}{2}\right)\beta\right]\sin\frac{n\alpha}{2}}{\sin\frac{\beta}{2}} \qquad \ldots\ldots\ldots\ldots \text{(B')}$$

In the above results it should be noted that

$$\sin\frac{\alpha}{2} \neq 0 \text{ or } \alpha \neq 2\,r\,\pi, \text{ r is an integer.}$$

If in the results A and B, we take $\beta = \frac{2\pi}{n}$

then

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \ldots\ldots\ldots + \cos\left(\alpha + \frac{2(n-1)\,\pi}{n}\right) = 0$$

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \ldots\ldots\ldots + \sin\left(\alpha + \frac{2(n-1)}{n}\alpha\right) = 0$$

1.3.3 Some Important Examples.

Example 1 : Evaluate

$$\sin^4\alpha + \sin^4(\alpha+\beta) + \ldots\ldots\ldots + \sin^4\left(\alpha + \overline{n-1}\,\beta\right)$$

Solution : First of all we will have to express

$\sin^4\theta$ in terms of multiples of $\cos\theta$ and $\sin\theta$.

Now

z

$$z = \cos\theta + i\sin\theta \quad \frac{1}{z} = \cos\theta - i\sin\theta$$

i

$$= \qquad Z - \frac{1}{2} = 2\,i\sin\theta$$

$$= \qquad (2\sin\theta)^4 = \left(Z - \frac{1}{z}\right)^4$$

$$= Z^4 + {}^4C_1\, Z^2\; {}^4C_2 -{}^4 C_3 \cdot \frac{1}{z^2} +{}^4 C_4 \cdot \frac{1}{z^4}$$

$$= \left(Z^4 + \frac{1}{z^4}\right) - {}^4C_1 \left(Z^2 + \frac{1}{z^2}\right) + {}^4C_2$$

$$= \left(Z^4 + \frac{1}{z^4}\right) - 4\left(Z^2 + \frac{1}{z^2}\right) + 6$$

$= 2 \cos 4\theta - 4.2 \cos 2\theta + 6$

$16 \sin^4 \theta = 2 \cos 4\theta - 8 \cos 2\, \theta + 6$

$$\sin^4 \theta = \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$$

Therefore

$$\sum_{m=0}^{n-1} \sin^4 (\alpha + m\beta) = \frac{1}{8}\left[\{\cos 4\alpha - 4\cos 2\alpha + 3\}\right]$$

$$+ \{\cos 4(\alpha - \beta) - 4\cos 2(\alpha + \beta) + 3\} + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \left\{\cos 4\left(\alpha + \overline{n-1}\,\beta\right) - 4\cos 2\left(\alpha + \overline{n-1}\,\beta + 3\right)\right\}$$

$$= \qquad \sum_{m=0}^{n-1} \sin^4 (\alpha + m\beta)$$

$$= \frac{1}{8}\left[\left\{\cos 4\alpha + \cos 4(\alpha + \beta) + \ldots\ldots\ldots + \cos 4\left(\alpha + \overline{n-1}\right)\right\}\right.$$

$$\left. -4\left\{\cos 2\alpha + \cos 2(\alpha + \beta) + \ldots\ldots\ldots + \cos 2\left(\alpha + \overline{n-1}\,\beta\right) + 3n\right\}\right]$$

Using the result (A)

$$= \frac{1}{8}\left[\frac{\cos (4\alpha + 2(n-1)\beta) \sin 2n\beta}{\sin 2B} - 4\frac{\cos (2\alpha + \overline{n-1}\,\beta) \sin n\beta}{\sin B} + 3n\right]$$

Example 2 : Evaluate

$$\sum_{k=1}^{n} = 1 \sin kx \cos 3kx$$

Solution : Now $\sin kx \cos 3kx = \dfrac{1}{2}\left[\sin 4kx - \sin 2kx\right]$

$$\sum_{k=1}^{n} \sin kx \cos 3kx = \frac{1}{2}\sum_{k=1}^{n}(\sin 4kx - \sin 2kx)$$

$$= \frac{1}{2}\,\frac{\sin 2\,(n+1)\,x \sin 2nx}{\sin 2x} - \frac{\sin (1+n)\,x \sin nx}{\sin x}$$

Self Check Exercise

1. Sum upto n term

   $3 \sin \alpha + 5 \sin 2\alpha + 7 \sin 3\alpha + \ldots\ldots\ldots\ldots$

   ....

   ....

   ....

1.3.4 C + iS Method

$$C = \cos \alpha + x\cos(\alpha+\beta) + x^2 \cos(\alpha+2\beta) + \ldots\ldots\ldots + x^{n-1} \cos\left(\alpha + \overline{n-1}\,\beta\right)$$

$$S = \sin \alpha + x\sin(\alpha+\beta) + x^2 \sin(\alpha+2\beta) + \ldots\ldots\ldots + x^{n-1} \sin\left(\alpha + n - 1\,\beta\right)$$

Multiply the second expression by i and adding with first, we have

$C + iS = (\cos \alpha + i \sin \beta) + x\,[\cos(\alpha+\beta) + i \sin(\alpha+\beta)]$

where $\cos \alpha + i \sin \beta = C + iS$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + x^2\, e^{i(\alpha+2\beta)} + \ldots\ldots\ldots\ldots + x^{n-1}\, e^{i\left(\alpha+\overline{n-1}\,\beta\right)}$$

$$= e^{\alpha}\left[1 + xe^{i\beta} + x^2\, e^{2i\beta} + \ldots\ldots\ldots + x^{n-1}e^{i(n-1)\beta}\right]$$

Summing the terms within square brackets by G.P. $\begin{pmatrix} a = 1 \\ r = xe^{i\beta} \end{pmatrix}$

$$= e^{i\alpha}\,\frac{1\left(1 - x^n e^{in\beta}\right)}{1 - x\,e^{i\beta}} = \frac{e^{i\alpha}\left(1 - x^n e^{in\beta}\right)}{1 - x\,e^{i\beta}} \qquad \ldots\ldots\ldots\ldots (1)$$

and in a similar manner.

$$(C - iS) = e^{i\alpha}\,\frac{\left[1 - x^n e^{in\beta}\right]}{\left[1 - x\,e^{i\beta}\right]} \qquad \ldots\ldots\ldots\ldots (2)$$

Now

$$C + iS = \frac{e^{i\alpha}\left(1 - x^n e^{in\beta}\right)}{\left(1 - x\,e^{i\beta}\right)} = \frac{\left(1 - xe^{-i\beta}\right)}{\left(1 - xe^{i\beta}\right)}$$

$$= e^{-i\alpha}\,\frac{\left[1 - x^n e^{in\beta} - xe^{-in\beta} + x^{n+1}e^{-1(n-1)\beta}\right]}{1 - x\,e^{i\beta} - x\,e^{-i\beta} + x^3}$$

$$= \frac{e^{-i\alpha}\left[1 - x^n e^{in\beta} - x\,e^{-i\beta} + x^{n+1}e^{-i(n-1)\beta}\right]}{\left(1 - 2x\cos\beta + x^2\right)}\left(e^{i\beta} + e^{-i\beta} = 2\cos\beta\right) \qquad \text{............... (3)}$$

and

$$C - iS = \frac{e^{-i\alpha}\left[1 - x^n e^{-in\beta}\right]}{1 - x\,e^{-in\beta}} \times \frac{1 - x\,e^{i\beta}}{1 - x\,e^{-i\beta}}$$

$$= \frac{e^{-i\alpha}\left[1 - x^n e^{-in\beta} - xe^{i\beta} + x^{n+1}e^{-i(n-1)\beta}\right]}{\left(1 - 2x\cos\beta + x^2\right)} \qquad \text{............ (4)}$$

On adding the subtracting, we get the values of C and S.

Example 3 : Sum the series $\tan z + 2\tan 2z + 2^2 \tan 2^2 z + \ldots\ldots\ldots\ldots\ldots + 2^{n-1} \tan 2^{n-1} z$

Solution : nth term = $2^{n-1} \tan 2^{n-1} z$

Now,we know that

$\tan \alpha = \cot \alpha - 2 \cot 2\alpha$

$\therefore \quad U_n$ = nth term= $2^{n-1} [\cot 2^{n-1}z - 2 \cot 2^n z]$

$U_n = 2^{n-1} \cot 2^{n-1} Z - 2^n \cot 2^n z$

Putting n = 1, 2, ...

$U_1 = \cot z - 2\cot 2Z$

$U_2 = 2 \cot 2z - 2^2 \cot 2^2 Z$

.

.

.

$U_n = 2^{n-1} \cot 2^{n-1} z - 2^n \cot 2^n Z$

On adding

$$\sum U_n\ U_n = \cot z - 2^n \cot 2^n z$$

Example 4: $\sin \alpha \sin 2\alpha + \sin 2\alpha \sin 3\alpha + \ldots\ldots\ldots\ldots\ldots$ n terms

Solution : Now $\sin\alpha \sin 2\alpha = \frac{1}{2}[\cos\alpha - \cos 3\alpha]$

$$U_1 = \frac{1}{2}(\cos\alpha - \cos 3\alpha)$$

$$U_2 = \frac{1}{2}(\cos 3\alpha - \cos 5\alpha)$$

$$U_3 = \frac{1}{2}(\cos 5\alpha - \cos 7\alpha)$$

.

.

.

$$U_n = \frac{1}{2}\left[\cos(2n-1)\alpha - \cos(2n+1)\alpha\right]$$

On adding

$$\sum U_n = \frac{1}{2}\left[\cos(2^{n-1})\alpha - \cos(2n+1)\alpha\right]$$

$$\frac{1}{2}.2\sin(n+1)\alpha \sin n\alpha$$

$$= \sin(n+1)\alpha \sin n\alpha$$

1.3.5 Hyperbolic Functions of a Complex Variable :

$$\sin hz = \frac{e^z - e^{-z}}{2} \qquad [-ve]$$

$$\cos hz = \frac{e^z - e^{-z}}{2} \qquad [+ve]$$

$$\tan hz = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Or

$$\sin i\, z = \frac{e^{-z} - e^{z}}{2i}$$

$$\frac{e^{iz} - e^{-z}}{2i} = \frac{ie^{z} - e^{-z}}{2}$$

sin iz = i sinh z .... (1)

Similarly $$i \sin 2z = \frac{2\, i \tanh z}{1 - \tanh^2 Z}$$

$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{z} - e^{x}}{2}$$

cos i z = i cosh z .... (1)

Also tan iz = i tanh z

We also know that

$$\sin 2Z = \frac{2 \tan z}{1 + \tan^2 z}$$

$$\sin 2iZ = \frac{2 \tan iz}{1 + \tan^2 iz}$$

$$i \sin h\, 2Z = \frac{2i \tan iz}{1 - \tanh^2 iz}$$

$$\sinh 2Z = \frac{2 \tanh z}{1 - \tanh^2 z} = \frac{2 \tanh z}{1 - \tanh^2 z}$$

Example 5 : If Z = x + iy express tan Z as A + iB

Solution : $\tan Z = \tan (x + iy) = \dfrac{e^{i(x-iy)} - e^{-i(x-iy)}}{\left[e^{i(x+iy)} + e^{-i(x-iy)}\right]1}$

$$= \frac{-i\left[e^{ix}.e^{-y} - e^{ix}.e^{y}\right]}{\left[e^{ix}.e^{-y} - e^{ix}.e^{y}\right]}$$

$$= \frac{i\left[e^{2ix-2y} - 1\right]}{e^{2ix-2y} + 1}$$

$$= \frac{-i\left[e^{2ix-2y} - 1\right]}{e^{2ix-2y} + 1} \cdot \left[\frac{\left[1 + e^{2ix-2y}\right]}{1 + e^{2ix-2y}}\right]$$

$$= \frac{-i\left[2^{2ix-2y} - 1 + e^{4y} - e^{2ix-2y}\right]}{1 + e^{2ix-2y} + e^{-4y} + e^{-2ix-2y}}$$

$$= \frac{-i\left[-1 + e^{-4y} + e^{-2y}\left(e^{2ix} - e^{-2ix}\right)\right]}{\left[1 + e^{-4y} + \left(e^{2ix} - e^{-2ix}\right)e^{-2y}\right]}$$

Multiply numerator and denominator by e^{2y}

$$= \frac{-i\left[-e^{2y} + e^{-2y} + 2i\sin 2x\right]}{\left[e^{2y} + e^{-2y} + 2\cos 2x\right]}$$

$$= \frac{-i\left[-2\sinh 2y + 2i\sin 2x\right]}{\left[2\cosh 2y + 2\cos 2x\right]}$$

$$= \frac{\left[= 1\sinh 2y + \sin 2x\right]}{\left[\cosh 2y + \cos 2x\right]}$$

$$= \frac{\sin 2x + i\sinh 2x}{\cos 2x + \cosh 2y} = \frac{\sin 2x}{\cos 2x - \cosh 2y} + i\,\frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

1.3.6 Summary

In this lesson, we have learnt to evaluate the sum of finite trigonometric series. We have also introduced hyperbolic functions of a complex variable.

1.3.7 Key Concepts

Summation of finite trigonometric series, hyperbolic functions of a complex variable.

1.3.8 Long Questions

1. If sin (U + iv) =x + iy, prove

(i) $$\frac{x^2}{\cosh^2 v}+\frac{y^2}{\cosh^2 v}=I, V\neq 0$$

(ii) $$\frac{x^2}{\cos n^2 U}+\frac{y^2}{\cos^2 U}=I, U\neq n\frac{n}{2},\ \text{n any integer.}$$

2. If tan (A + iB) = x + iy, then

$x^2 + y^2 + 2x \cot 2A = 1$

$x^2 + y^2 - 2y \cot h\, 2B = 1$

3. Sum

$$\sinh\theta + n\sinh 2\theta + \frac{n(n-1)}{2!}\sinh 3\theta + \ldots\ldots\ldots\ldots (n+1) \text{ terms}$$

4. Find the sum of n terms of the following series :

(i) cosec α + cosec 2α + cosec 3α +.............

(ii) cosec θ + cosec 4θ + cosec 4θ cosec 7θ +.............

(iii) $\cos^3\alpha + \cos^3 2\alpha + \cos^3\alpha$ +..........

(iv) $$\sin\alpha + \sin\left(\alpha+\frac{2\pi}{n}\right)+\sin\left(\alpha+\frac{2\pi}{n}\right)+\ldots\ldots\ldots$$

5. Show that

$$\tan n\theta = \frac{\sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots\ldots n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots. n \text{ terms}}$$

6. Sum the following series upto n terms.

(a) sinh a + sinh (a + b) + sinh (a + 2b) + ...

(b) cosh a + cosh (a + b) + cos (a + 2b) + ...

1.3.9 Short Questions

1. Separate into real and imaginary parts :

(i) sin (x + iy); x, y real.

(ii) cosh (x + iy); x, y real.

1.3.10 Suggested Readings

R.S. Verma and K.S. Shukla, Text book on Trigonometry, Pothishala Pvt. Ltd., Allahabad

Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.