Centre for Distance and Online Education

Punjabi University, Patiala

Class : B.A. I (Mathematics) **Semester : I**

Paper : MTHB1101T (Calculus) **Unit-1**

Medium : English

Lesson No.

Website : www.pbidde.org

MTHB1101T: CALCULUS

Course Outcomes:	
CO1	To understand the order completeness properties of real numbers
CO2	Able to learn basic properties of limits, infinite limits, indeterminate forms.
CO3	To understand Continuous functions, types of discontinuities, continuity of composite functions.
CO4	To know Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, their geometric interpretation and applications.
CO5	To understand Hyperbolic, inverse hyperbolic functions of a real variable and their derivatives.

For Regular Students / Students of Centre for Distance and Online Education
Maximum Marks: 50 Marks

Maximum Time: 3 Hrs.

For Regular students:6Lectures of 45minutes/week

External Marks: 35
Internal Assessment: 15
Pass Percentage: 35%
For Private Students
Maximum Marks: 50 Marks

INSTRUCTIONS FOR THE PAPER-SETTER

The question paper will consist of three sections A, B and C. Sections A and B will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eleven short answer type questions covering the entire syllabus uniformly. Each question in Sections A and B will be of 06 marks and Section C will be of 11 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Sections A and B and compulsory question of Section C.

SECTION-A

Properties of real numbers :
Order property of real numbers, bounds, l.u.b. and g.l.b. order completeness property of real numbers, archimedian property of real numbers.
Limits: ε -δ definition of the limit of a function, basic properties of limits, infinite limits, indeterminate forms.
Continuity: Continuous functions, types of discontinuities, continuity of composite functions, continuity of f (x) , sign of a function in a neighborhood of a point of continuity, intermediate value theorem, maximum and minimum value theorem.

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SECTION-B

Mean value theorems: Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, their geometric interpretation and applications, Taylor's theorem, Maclaurin's theorem with various form of remainders and their applications.

Hyperbolic, inverse hyperbolic functions of a real variable and their derivatives, successive differentiations, Leibnitz's theorem.

REFERENCES :

1. J. D. Murray & M . R. Spiegel : Theory and Problems of Advanced Calculus, Schaum's Outline Series, Schaum Publishing Co., New York.

2. P.K. Jain and S. K. Kaushik : An Introduction to Real Analysis, S. Chand & Co., New Delhi, 2000.

3. Gorakh Prasad : Differential Calculus, Pothishala Private Ltd., Allahabad.

4. G.B. Thomas & R.L. Finney : Calculus and Analytic Geometry (Ninth Edition), Pearson Publication.

5.Shanti Narayan and P.K. Mittal : Differential Calculus, Edition 2006, S. Chand & Co., New Delhi.

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B.A. I (Mathematics) **Semester: 1**
PAPER : MTHB1101T **Unit-I**
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LESSON NO. 1.1 **AUTHOR : Dr. Chanchal**
Last Updated: July 2024

REAL NUMBER SYSTEM

1.1.1 Objectives

After going through this lesson, the students will be able

- to have the knowledge of field of real numbers and its important properties;
- to understand the proofs of various theorems;
- to define absolute value or modulus of a real number;

- to understand the concept of l.u.b. and g.l.b.
- to study Order Completeness Property and Archimedean Property of real numbers.

1.1.2 Introduction

We are already familiar with the following concepts:

- The set $N = \{1, 2, 3, \ldots\}$ consisting of natural numbers $1, 2, 3....$, is called the set of natural numbers.
- The set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ consisting of integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$, is known as the set of integers.
- Any number of the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, is called a rational number. In set builder form, the set of rational numbers (Q) is given by $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$. Every rational number can be expressed as a terminating or a recurring decimal expression
- Any number which is not a rational number is called an irrational number. For example: $\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots$ are irrational numbers. Every irrational number has a non-terminating and non-repeating decimal expression.
- The union of set of rational numbers and set of irrational numbers is known as the set of real numbers and it is denoted by $\mathbb{R}$.

1.1.3 Field of Real Numbers

The set of real numbers $\mathbb{R}$ forms a field under the operations of addition and multiplication as it satisfies the following properties:

Properties Under Addition:

1. Closure Property: Addition of two real numbers is a real number i.e. $\forall a, b \in \mathbb{R}, a + b \in \mathbb{R}$.
2. Commutative Property: $\forall a, b \in \mathbb{R}, a + b = b + a$.

3. Associative Property: $\forall a, b, c \in \mathbb{R}, (a+b)+c = a+(b+c)$.
4. Existence of Additive Identity: There exists an element 0 in $\mathbb{R}$ such that $a+0 = 0+a = a, \forall a \in \mathbb{R}$, where 0 is called the additive identity.
5. Existence of Additive Inverse: There exists an element $-a$ in $\mathbb{R}$ such that $a+(-a) = (-a)+a = 0, \forall a \in \mathbb{R}$. Here, $-a$ is known as the additive inverse of a in $\mathbb{R}$.

Properties Under Multiplication:

1. Closure Property: Product of two real numbers is again a real number i.e. $\forall a, b \in \mathbb{R}, a \cdot b \in \mathbb{R}$.
2. Commutative Property: $\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$.
3. Associative Property: $\forall a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
4. Existence of Additive Identity: There exists an element 1 in $\mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a, \forall a \in \mathbb{R}$, where 1 is called the multiplicative identity.

   Note: The multiplicative identity element 1 is unique.
5. Existence of Additive Inverse:$\forall a \neq 0 \in \mathbb{R}$, there exists an element $\frac{1}{a}$ (or a^{-1}) in $\mathbb{R}$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$. Here, $\frac{1}{a}$ (or a^{-1}) is known as the multiplicative inverse of $a \neq 0$ in $\mathbb{R}$.

Distributive Property

Multiplication is distributive over addition in $\mathbb{R}$ i.e.

$$\forall a, b, c \in R, a \cdot (b+c) = a \cdot b + a \cdot c$$

1.1.4 Important Results

The below stated results/properties (easy to prove) hold good for the field of real numbers:

a. The additive identity element 0 is unique.

b. The additive inverse $(-a)$ of every real number a in $\mathbb{R}$ is unique.

c. The multiplicative identity element 1 is unique.

d. The multiplicative inverse $(\frac{1}{a})$ of every real number $a \neq 0$ in $\mathbb{R}$ is unique.

e. **Cancellation Laws:** For Addition: If $a+c = b+c$, then $a = b$, $\forall a, b, c \in \mathbb{R}$.

For Multiplication: If $a.c = b.c$, then $a = b$, $\forall a, b, c \in \mathbb{R}$ (Here, $c \neq 0$).

f. If $a.b = 0$ for $a, b \in \mathbb{R}$, then either $a = 0$ or $b = 0$.

g. $-(-a) = a \ \forall a \in \mathbb{R}$,
$\frac{1}{\frac{1}{a}} = a \ \forall$ non-zero $a \in \mathbb{R}$,
$(-1)(a) = -a \ \forall a \in \mathbb{R}$.

h. $\forall a, b \in \mathbb{R}$, $-(a+b) = (-a) + (-b)$,
$a(-b) = (-a)b = -ab$ and $(-a)(-b) = ab$.

i. **Transitive Law:** For a,b,c $\in \mathbb{R}$, If $a < b, b < c$, then $a < c$ and if $a > b, b > c$, then $a > c$.

j. **Trichotomy Law**: For $a, b \in \mathbb{R}$, one and only one of the following holds :

(i) $a < b$,

(ii) $a = b$,

(iii) $a > b$.

Theorem 1 (Statement): If a, b are two real numbers, then their arithmetic mean $\frac{a+b}{2}$ is also a real number and it lies between them.

Proof: Let us assume that $a < b$ (using Trichotomy law $a, b \in \mathbb{R}$)

As $a, b \in \mathbb{R}$, therefore, $a + b$ and hence $\frac{a+b}{2}$ is also a real number.

Further, $a < b \implies a + a < a + b \implies 2a < a + b$

Also $a < b \implies a + b < b + b \implies a + b < 2b$

On combining the above two results,

$$2a < a+b < 2b \implies a < \frac{a+b}{2} < b$$

Hence, the arithmetic mean of two real numbers is a real number and it lies between them.

Corollary: There exists infinitely many real numbers between any two distinct real numbers.

Proof: Let $a, b \in \mathbb{R}$ and $a < b$.

By using the above Theorem 1,

$$a < \frac{a+b}{2} < b \text{ i.e. real number } \frac{a+b}{2} \text{ lies between } a \text{ and } b.$$

Using the Theorem 1 again for $a, \frac{a+b}{2} \in \mathbb{R}$ and $\frac{a+b}{2}, b \in \mathbb{R}$ respectively, we obtain

$$a < \frac{a + \frac{a+b}{2}}{2} < \frac{a+b}{2}$$

and

$$\frac{a+b}{2} < \frac{\frac{a+b}{2} + b}{2} < b$$

On combining the above two results, we obtain

$$a < \frac{a + \frac{a+b}{2}}{2} < \frac{a+b}{2} < \frac{\frac{a+b}{2} + b}{2} < b$$

$$\implies a < \frac{3a+b}{4} < \frac{a+b}{2} < \frac{a+3b}{4} < b$$

$$\implies \text{Three real numbers } \frac{3a+b}{4}, \frac{a+b}{2}, \text{ and } \frac{a+3b}{4} \text{ lie between } a \text{ and } b.$$

Continuing in the same manner, we can state that an infinite number of real numbers lie between a and b.

Theorem 2 (Statement): Let a, b be the positive real numbers and $a < b$. Then, prove that their geometric mean lies between them i.e. $a < \sqrt{ab} < b$.

Proof: We have, $b > a > 0$.

Now $b > a \implies b \cdot a > a \cdot a$

$$\implies ba > a^2 \implies \sqrt{ab} > a \quad ....(1)$$

Again $b > a$

$$\implies b \cdot b > b \cdot a \implies b^2 > ab$$

$$\implies b > \sqrt{ab} \quad ....(2)$$

From (1) and (2), we get $a < \sqrt{ab} < b$.

Self Check Exercise: For $a, b \in \mathbb{R}$ with $a < b$, prove that $a < a + \frac{b-a}{n} < b$, where $n \in \mathbb{N}$ and $n > 1$.

Solution : ....

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1.1.5 Absolute Value of a Real Number

The absolute value or modulus or numerical value of a real number x (denoted by $|x|$) is defined as follows:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \max(x, -x) \quad \text{or} \quad \sqrt{x^2}$$

Note: For $x \in \mathbb{R}$

(i) $|x| \geq 0$

(ii) $|x| = |-x|$

(iii) $|x^2| = x^2$

(iv) $-|x| \leq x \leq |x|$

Theorem 3 (Statement): If $a, b \in \mathbb{R}$, then

(i) $|ab| = |a||b|$

(ii) $|\frac{a}{b}| = \frac{|a|}{|b|}, b \neq 0$

(iii) $|a| = |b| \Leftrightarrow a = \pm b$

(iv) Triangular Inequality: $|a + b| \leq |a| + |b|$. Moreover, the equality holds if $ab \geq 0$ i.e. a and b are of same sign.

(v) $|a-b| \geq ||a|-|b||$

Proof. Since $|x| = \sqrt{x^2}$

(i)

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$$

(ii)

$$|\frac{a}{b}| = \sqrt{\left(\frac{a}{b}\right)^2} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}, b \neq 0$$

(iii)

$$|a| = |b| \Leftrightarrow |a|^2 = |b|^2 \Leftrightarrow a^2 = b^2 \Leftrightarrow a = \pm b$$

(iv)

$$|a+b|^2 = (a+b)^2 = a^2 + b^2 + 2ab \leq |a|^2 + |b|^2 + 2|a||b|$$

$$= (|a| + |b|)^2$$

$$\therefore |a+b| \leq |a| + |b|$$

$$\text{Now, } |a+b| = |a| + |b| \text{ iff } |a+b|^2 = (|a|+|b|)^2$$

$$\text{iff } (a+b)^2 = |a|^2 + |b|^2 + 2|a||b|$$

$$\text{iff } a^2 + b^2 + 2ab = a^2 + b^2 + 2|ab|$$

$$\text{iff } ab = |ab|$$

$$\text{iff } ab \geq 0$$

(v)

$$|a| = |a - b + b| \leq |a-b| + |b| \quad \text{(By triangular inequality)}$$

$$\implies |a| - |b| \leq |a-b|$$

or

$$|a-b| \geq |a| - |b| \quad ....(1)$$

Interchanging 'a' and 'b' in (1), we have

$$|b-a| \geq |b| - |a|$$

$$\implies |a-b| \geq -(|a|-|b|) \quad ....(2)\ (\text{As}|-a| = |a|)$$

From (1) and (2), we get

$$|a-b| \geq ||a|-|b||$$

Self Check Exercise: For $a, b \in \mathbb{R}$, prove that $|a-b| \leq |a| + |b|$.

Solution : ....

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Theorem 4 (Statement): Let a be a positive real number, then

(i) $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$,

(ii) $|x| > a \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$.

Proof: If $x \geq 0$, $|x| = x$

$$\therefore |x| < a \Leftrightarrow 0 \leq x < a$$

$$\text{and if } x < 0, |x| = -x \text{ and } -x > 0 \quad ....(1)$$

$$\therefore |x| < a \Leftrightarrow 0 < -x < a \Leftrightarrow -a < x < 0 \quad ....(2)$$

Combining (1) and (2), we have

$$|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$$

(ii)

$$\text{If} x \geq 0, \text{then} |x| = x$$

$$\therefore |x| > a \Leftrightarrow x > a \text{ ....(1)}$$

$$\text{If} x < 0, \text{then} |x| = -x$$

$$\therefore |x| > a \Leftrightarrow -x > a \Leftrightarrow x < -a \text{ ....(2)}$$

Combining (1) and (2), we get

$$|x| > a \Leftrightarrow x < -a \text{ or } x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$$

1.1.6 Least Upper Bound and Greatest Lower Bound

To understand the concept of least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of a set, we firstly study about bounds i.e. lower bound and upper bound as defined below:

Let S be a non-empty subset of real numbers.

(i) If there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$, then M is called an upper bound of S. If the set S has an upper bound, then S is said to be bounded above.

(ii) If there exists $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in S$, then m is called a lower bound of S. If S has a lower bound, then S is said to be bounded below.

Bounded and Unbounded Sets:

A set S is said to be bounded if it is bounded below as well as bounded above, i.e., S is bounded if there exists $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in S$. In other words, a set $S \neq 0$ is said to be bounded if there exists a positive real number N such that $|x| \leq N$ for all $x \in S$. If the set S it is not bounded above or not bounded below or neither bounded above nor bounded below, then it is said to be an unbounded set.

For example:

(i) Every finite interval say (a, b) is a bounded set as it is bounded below by a and bounded above by b.

(ii) The set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is bounded below by 1 but it is not bounded above.

(iii) The interval $(-\infty, 1)$ is bounded above by 1 but is not bounded below.

(iv) $\mathbb{Z}$, the set of integers, is neither bounded above nor bounded below.

Note: If M is an upper bound of the set S, then every number greater than M is also an upper bound of S. Thus, if S is a set which is bounded above, then it has infinitely many upper bounds. Also, if m is a lower bound of S, then every number less than m is also a lower bound of S. In other words, If the set S is bounded below, then it has infinitely many lower bounds.

Definition (l.u.b.): Let S be a bounded above subset of $\mathbb{R}$. Then $u \in \mathbb{R}$ is called the least upper bound (**l.u.b.**) of S or *supremum* of S ($\sup S$) if

(i) $u \geq x$ for all $x \in S$ i.e., u is an upper bound of S.

(ii) If u' is any upper bound of S then $u \leq u'$.

Definition (g.l.b.): Let S be a bounded below subset of $\mathbb{R}$. Then $l \in \mathbb{R}$ is called the greatest lower bound (**g.l.b.**) of S or *infimum* of S ($\inf S$) if

(i) $l \leq x$ for all $x \in S$ i.e., l is a lower bound of S.

(ii) If l' is any lower bound of S, then $l \geq l'$.

Remarks

1. l.u.b. and g.l.b. of a set may or may not belong to the set. For example: If $A = [1, 2]$, then l.u.b. $S = 2$ and $2 \in S$ but if $A = [1, 2)$, then l.u.b. $S = 2$ and $2 \notin S$.

2. The l.u.b. (if exists) and g.l.b. (if exists) of a set are unique.

1.1.7 Order Completeness Property of Real Numbers

This property states that every non-empty subset of real numbers which is bounded above has a l.u.b. in $\mathbb{R}$. This property is also known as Least Upper Bound Property of real numbers.

Theorem 5 (Statement): The set of rational numbers $\mathbb{Q}$ does not possess order completeness property.

Proof: To prove the above result, we prove that there exists a non-empty subset Q^* of rational numbers which is bounded above but does not have a least upper bound.

Let $Q^* = \{x : x \in \mathbb{Q} \text{ and } x^2 < 3\}$.

Clearly $1 \in Q^* \Rightarrow Q^* \neq \phi$

Further, if α is any positive number such that $\alpha^2 > 3$, then α is an upper bound of Q^*. Thus Q^* is a non-empty set of rational numbers which is bounded above. Now, we have to prove that Q^* has no least upper bound in $\mathbb{Q}$.

If possible, let $a \in \mathbb{Q}$ be the l.u.b. of Q^*. Clearly a is positive and one and only one of the following (i) $a^2 < 3$, (ii) $a^2 = 3$, (iii) $a^2 > 3$ holds.

Let $b = \frac{2a+3}{a+2}$. Clearly, b is a positive rational number as a is a positive rational number. Now, we discuss the following three cases:

- **Case I For $a^2 < 3$:**

  Here, $b^2 - 3 = \left(\frac{2a+3}{a+2}\right)^2 - 3 = \frac{-(3-a^2)}{(a+2)^2} < 0$ showing that $b \in Q^*$.

  Now, $b - a = \frac{2a+3}{a+2} - a = \frac{3-a^2}{a+2} > 0$ which implies $b > a$

  This contradicts the result that a is an upper bound of Q^*.

  $\therefore$ Our supposition is wrong.

- **Case II For $a^2 = 3$:**

  Since a is positive, $\therefore a = \sqrt{3}$ and $\sqrt{3}$ is not a rational number, which is not possible.

- **Case III For $a^2 > 3$:**

Here, $b^2 - 3 = \left(\frac{2a+3}{a+2}\right)^2 - 3 = \frac{a^2-3}{(a+2)^2} > 0$ showing that b is an upper bound of Q^*.

Now, $b - a = \frac{2a+3}{a+2} - a = \frac{-(a^2-3)}{a+2} < 0$ which implies $b < a$

Thus we obtain an upper bound b of Q^* such that $b < a$ which is again a contradiction.

From all the three cases, it follows that Q^* has no l.u.b. in $\mathbb{Q}$.
Hence the set of rational number is not a completely ordered field.

Theorem 6 (Statement): Let S be a non-empty subset of $\mathbb{R}$ and S is bounded above. Then, $u \in \mathbb{R}$ is the l.u.b. S iff

(i) u is an upper bound of S,

(ii) $\forall \epsilon > 0$, however small, there exists at least one $x \in S$ such that $x > u - \epsilon$.

Proof: Necessart Part: u is an upper bound of S (since u is the l.u.b. of S). Moreover, $\forall \epsilon > 0$, however small, $u - \epsilon < u$.

$$\Rightarrow u - \epsilon \text{ is not an upper bound of } S.$$

$$\Rightarrow \text{there exists at least one } x \in S \text{ such that } x > u - \epsilon.$$

Therefore, both the conditions (i) and (ii) hold.

Sufficient Part: It is given that $u \in \mathbb{R}$ satisfies the conditions (i) and (ii). We need to prove that u is the l.u.b. of S. If possible, let $v(< u) \in \mathbb{R}$ be any other upper bound of S.

Set $\epsilon = u - v > 0$. By (ii), there exists at least one $x \in S$ such that

$$x > u - \epsilon \Rightarrow x > u - (u - v) \Rightarrow x > v.$$

Therefore, v is not an upper bound of S.
Hence, any real number less than u cannot be an upper bound of S or we can say that u is the l.u.b. of S.

On similar lines, the below stated theorem can be proved very easily.

Theorem (Statement): Let S be a non-empty subset of $\mathbb{R}$ and S is bounded below. Then, $l \in \mathbb{R}$ is the g.l.b. of S iff

(i) l is a lower bound of S, and

(ii) $\forall \epsilon > 0$, however small, there exists at least one $x \in S$ such that $x < l + \epsilon$.

1.1.8 Archimedean Property of Real Numbers

Theorem 7 (Statement): For given $a > 0$ and $b \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $na > b$.

Proof. Since b is any real number, therefore, either $b > 0$ or $b = 0$ or $b < 0$.

Case I. When $b \leq 0$, then $na > b$ holds $\forall n \in \mathbb{N}$ $\quad$ [na is positive]

Case II. When $b > 0$. Suppose the result is false i.e. $na \leq b$ for any $n \in \mathbb{N} \implies na \leq b \forall n \in \mathbb{N}$.

Thus, the set $S = \{na, n \in \mathbb{N}\}$ is bounded above by b.

By the order completeness property of real numbers, the set S must have a l.u.b., say b'

$$\Rightarrow na \leq b' \forall n \in \mathbb{N}.$$

$$\Rightarrow (n+1)a \leq b' \forall n \in \mathbb{N}.$$

$$\Rightarrow na \leq b' - a, \forall n \in \mathbb{N}.$$

$$\Rightarrow b' - a \text{ is an upper bound of } S.$$

Thus, a number $b' - a$, strictly less than the l.u.b. (b') is an upper bound of S, which is a contradiction. Therefore, our supposition is wrong. Hence, there exists a positive integer n, such that $na > b$, which proves the theorem.

Theorem 8 (Statement): For any real number x, there exists a unique integer m such that

$$m \leq x < m + 1.$$

Proof. Consider the set $S = \{n : n \in \mathbb{Z} and n \leq x\}$

Clearly, S is a non-empty subset of real numbers bounded above by x.

Now, by the order completeness property, S has a l.u.b., say m, where $m \in \mathbb{Z}$ and it is the greatest element of S.

∴m $\in S$ and $m + 1 S \implies m \leq x$ and $m + 1 > x$

$\implies m \leq x < m + 1$ and m is unique (since l.u.b. of a set is unique).

Hence, for any $x \in \mathbb{R}$, there exists a unique integer m such that $m \leq x < m + 1$.

1.1.9 Summary : Following concepts are summarized from this lesson:

- The set of real numbers $\mathbb{R}$ forms a field under the operations of addition and multiplication.
- For $a, b \in \mathbb{R}$, their arithmetic mean and geometric mean lies between them.
- There exists infinitely many real numbers between any two real numbers.
- $|x| = \max(x, -x)$ or $\sqrt{x^2}$ where $|x|$ means absolute value of x.
- Triangular Inequality: $|a + b| \leq |a| + |b|$ for $a, b \in \mathbb{R}$.
- A non-empty set is bounded if there exists a positive real number N such that $|x| \leq N$ for all $x \in S$.
- The l.u.b. (if exists) and g.l.b. (if exists) of a set are unique and these may or may not belong to the set.
- Every non-empty subset of real numbers which is bounded above has a l.u.b. in $\mathbb{R}$. This property is known as Order Completeness Property of Real Numbers.
- The set of rational numbers is not a completely ordered field.

1.1.10 Glossary : Field of Real Numbers; Properties of Real Numbers; Absolute Value Function; Lower and Upper Bounds; Bounded Set; g.l.b. and l.u.b. of a Set; Order Completeness Property of Real Numbers; Archimedean Property of Real Numbers.

1.1.11 Long Questions

1. Solve the inequality $3 + x \leq 5x - 2 \leq 7 + x$.
2. Solve the inequality $\frac{5x-6}{x+6} < 1$.
3. Solve the inequality $|x - 1| + |x + 2| = 3$.
4. Show that the set $A = \{\frac{4+x}{4-x} : x > 0, x \neq 4\}$ is neither bounded above nor bounded below.

1.2.10 Short Questions

1. Solve the inequality $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$.
2. Solve the following:

   a. $\frac{2}{|3-5x|} \geq 7$,

   b. $|1 + x| = -1$.
3. Prove that $\left|x - \frac{1}{2}\right| < \frac{1}{3}$ iff $\frac{1}{11} < \frac{1-x}{1+x} < \frac{5}{7}$.
4. Find l.u.b. and g.l.b. (if exists) for the following sets:

   a. $\{\frac{2+x}{2-x} : 0 \leq x \leq 1\}$,

   b. $\{\frac{n+1}{n} : n \in \mathbb{N}\}$.

1.2.11 Suggested Readings

1. Shanti Narayan and P.K. Mittal, Differential Calculus, S. Chand and Co., New Delhi (Edition 2006).
2. Gorakh Prasad, Differential Calculus, Pothishala Pvt. Ltd. Allahabad.

B.A. I (Mathematics) **Semester: 1**
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LESSON NO. 1.2 **AUTHOR : Dr. Chanchal**
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LIMIT AND CONTINUITY OF FUNCTIONS

1.1.1 Objectives

After going through this lesson, the students will be able

- to have the knowledge of neighbourhood of a point;
- to understand the concept of limit of a function;
- to state and prove Cauchy's Criterion for Limits;
- to understand the algebra of limits;

- to state and prove Squeeze Principle.

1.1.2 Introduction

Neighbourhood of a Point: If $a, b \in \mathbb{R}$ and $a < b$, then the set $\{x \in \mathbb{R} : a < x < b\}$ is called an open interval (a, b). Any open interval containing $c \in \mathbb{R}$ is known as the neighbourhood of point c i.e. if $c \in (a, b)$, then (a, b) is a neighbourhood of c. Particularly, if $|c-a| = |b-c|$, then (a, b) is called a symmetrical neighbourhood of the point c. Now, we can also define neighbourhood of c as:

For a given $\delta > 0$, the open interval $(c-\delta, c+\delta) = \{x \in \mathbb{R} : |x-c| < \delta\}$ (a symmetrical neighbourhood about c) is called the δ-neighbourhood of c.

Important Points:

1. If $a < c < b$ i.e. $c \in (a, b)$, then $(a, c) \cup (c, b)$ is called the deleted neighbourhood of c.
2. The set $\{x \in \mathbb{R} : 0 < |x-c| < \delta\} = (c-\delta, c) \cup (c, c+\delta)$ is a deleted δ-neighbourhood of c.

1.1.3 Limit of a Function

Definition (Limit): A function f is said to have a limit l as $x \to a$, written as $\lim_{x\to a} f(x) = l$, if for given $\epsilon > 0$, however small, there exists $\delta > 0$ (depending upon ϵ) such that

$$|f(x) - l| < \epsilon \text{ whenever } 0 < |x-a| < \delta.$$

In other words, $\lim_{x\to a} f(x) = l$, if for given $\epsilon > 0$, however small, there exists $\delta > 0$ such that if x lies in the deleted neighbourhood $(a-\delta, a) \cup (a, a+\delta)$ of a, then $f(x)$ must lie in the neighbourhood $(l-\epsilon, l+\epsilon)$ of l.

Definition (Left Hand Limit): A function f is said to have a left hand limit l as $x \to a^-$, written as $\lim_{x\to a^-} f(x) = l$, if for given $\epsilon > 0$, however small, there exists $\delta > 0$ (depending upon ϵ) such that

$$|f(x) - l| < \epsilon \text{ whenever } x \in (a-\delta, a).$$

Definition (Right Hand Limit): A function f is said to have a right hand limit l as $x \to a^+$, written as $\lim_{x\to a^+} f(x) = l$, if for given $\epsilon > 0$, however small, there exists $\delta > 0$ (depending upon ϵ) such that

$$|f(x) - l| < \epsilon \text{ whenever } x \in (a, a+\delta).$$

Note:

- Limit of a function may exist at the point a even if the function is not defined at a.
- Limit of a function exists iff both left hand limit and right hand limit exist and are equal. Mathematically, $\lim_{x\to a} f(x)$ exists iff both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal.

Theorem 1: Prove that $\lim_{x\to a} f(x)$ (if exists finitely) is unique.
Proof. If possible, suppose that

$$\lim_{x\to a} f(x) = l_1 \text{ and } \lim_{x\to a} f(x) = l_2 \text{ where } l_1 \neq l_2.$$

Without any loss of generality, we assume that $l_2 > l_1$. Let

$$|l_2 - f(x) + f(x) - l_1| > 0.$$

Now, $\lim_{x\to a} f(x) = l_1 \Rightarrow$ for given$\epsilon > 0$, however small, we can find a positive real number δ_1 (depending upon ϵ) such that

$$|f(x) - l_1| < \epsilon \text{ for } 0 < |x-a| < \delta_1 \quad \ldots (1)$$

Again $\lim_{x\to a} f(x) = l_2 \Rightarrow$ for given$\epsilon > 0$, however small, we can find a positive real number δ_2 (depending upon ϵ) such that

$$|f(x) - l_2| < \epsilon \text{ for } 0 < |x-a| < \delta_2 \quad \ldots (2)$$

Let $\delta = \min\ \{\delta_1, \delta_2\}$

Now from (1) and (2), we get

$$|f(x) - l_1| < \epsilon \text{ for } 0 < |x - a| < \delta \quad \ldots (3)$$

and

$$|f(x) - l_2| < \epsilon \text{ for } 0 < |x - a| < \delta \quad \ldots (4)$$

Further, $l_2 - l_1 = |l_2 - l_1| = |l_2 - f(x) + f(x) - l_1|$

$\leq |l_2 - f(x)| + |f(x) - l_1|$ (By triangle inequality)

$= |f(x) - l_2| + |f(x) - l_1|$

$< \epsilon + \epsilon$ (using (3) and (4))

$= 2\epsilon = l_2 - l_1 (\text{since} |l_2 - f(x) + f(x) - l_1|)$

$\therefore, l_2 - l_1 < l_2 - l_1$ which is a contradiction.

Hence, our supposition is wrong.

If $\lim_{x\to a} f(x)$ exists finitely, then it is unique.

Theorem 2: If $\lim_{x\to a} f(x) = l$, then $f(x)$ is bounded in the deleted neighborhood of a.

Proof: Since $\lim_{x\to a} f(x) = l$,

$$\therefore \text{for given } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$

$$|f(x) - l| < \epsilon \text{ whenever } 0 < |x - a| < \delta.$$

$$\therefore |f(x)| \leq |f(x) - l| + |l|$$

$$\leq \epsilon + |l|$$

$\therefore |f(x)| < M$, where $M = \epsilon + |l|$, whenever,

$$0 < |x - a| < \delta.$$

$$\Rightarrow f(x) \text{ is bounded in deleted neighbourhood of } a.$$

Theorem 3: If $\lim_{x\to a} f(x) = 0$ and $g(x)$ is bounded in a deleted neighbourhood of a, then

$$\lim_{x\to a} f(x)g(x) = 0$$

Proof: Since $g(x)$ is bounded in a deleted neighbourhood of a,

$$\therefore \text{there exists}\, 0 < M \in \mathbb{R}\, \text{such that}$$

$$|g(x)| \leq M \,\text{for}\, 0 < |x-a| < \delta_1, \ldots (1)$$

where δ_1 is some positive real number.

Further $\lim_{x\to a} f(x) = 0$,

$$\therefore \text{for given}\, \epsilon > 0,\, \text{there exists}\, \delta_2 > 0\, \text{such that}$$

$$|f(x)| < \frac{\epsilon}{M}\, \text{whenever}\, 0 < |x-a| < \delta_2 \quad \ldots (2).$$

Let $\delta = \min(\delta_1, \delta_2)$.

We have

$$|f(x)| < \frac{\epsilon}{M}\, \text{and}\, |g(x)| \leq M,\, \text{whenever}\, 0 < |x-a| < \delta.$$

$$\therefore |f(x)g(x)| = |f(x)||g(x)|$$

$$< \frac{\epsilon}{M} \cdot M = \epsilon\, \text{whenever}\, 0 < |x-a| < \delta.$$

$\therefore \lim_{x\to a} f(x)g(x) = 0$

1.2.4 Cauchy's Criterion for Limits

Theorem 4 (Statement): If $\lim_{x\to a} f(x)$ exists, then for given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $0 < |x_1 - a| < \delta$ and $0 < |x_2 - a| < \delta$.

Proof. Let $\lim_{x\to a} f(x) = l$ (exists finitely).

$\therefore$ For given $\epsilon > 0$, however small, there exists a positive real number δ such that

$$|f(x) - l| < \frac{\epsilon}{2} \text{ for } 0 < |x_1 - a| < \delta \quad \ldots (1)$$

$$|f(x) - l| < \frac{\epsilon}{2} \text{ for } 0 < |x_2 - a| < \delta \quad \ldots (2)$$

Now,

$$|f(x_1) - f(x_2)| = |f(x_1) - l + l - f(x_2)| = |(f(x_1) - l) - (f(x_2) - l)|$$

$$\leq |f(x_1) - l| + |f(x_2) - l|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } 0 < |x_1 - a| < \delta \text{ and } 0 < |x_2 - a| < \delta \quad [\text{ from (1) and (2)}]$$

Example 1: By using the definition of limit, prove the following

(i) $\lim_{x\to 2}(2 - 3x) = -4$

(ii) $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

Solution: (i) Let $f(x) = 2 - 3x$ and let $\epsilon > 0$, however small, be given.

Now,

$$|f(x) - (-4)| = |2 - 3x + 4| = |-3x + 6| = |3(x - 2)| = 3|x - 2|$$

$$\therefore |f(x) - (-4)| < \epsilon \text{ whenever } 0 < |x - 2| < \frac{\epsilon}{3}$$

i.e. $|f(x) - (-4)| < \epsilon$ whenever $0 < |x - 2| < \delta$, where $\delta = \frac{\epsilon}{3}$.

Thus for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - (-4)| < \epsilon \text{ whenever } 0 < |x - 2| < \delta.$$

Hence $\lim_{x\to 2} f(x) = -4$.

(ii) Let $f(x) = \frac{x^2-4}{x-2}$ and let $\epsilon > 0$, however small, be given.

$$\therefore |f(x) - 4| = \left|\frac{x^2 - 4}{x - 2} - 4\right| = \left|\frac{(x + 2)(x - 2)}{x - 2} - 4\right|$$

$$= |x - 2| < \epsilon \text{ whenever } 0 < |x - 2| < \epsilon$$

i.e. $|f(x) - 4| < \epsilon$ whenever $0 < |x - 2| < \delta$, where $\delta = \epsilon$.

Thus for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - 4| < \epsilon \text{ whenever } 0 < |x - 2| < \delta.$$

Hence,

$$\lim_{x \to 2} f(x) = 4$$

Example 2: Show that $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

Solution: Let $f(x) = \frac{1}{x-2}$

Let $0 < \epsilon < 1$ be given.

For any $\delta > 0$, Choose $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$ or $\frac{1}{n} < \delta$

Let $x_1 = 2 + \frac{1}{n}$ and $x_2 = 2 - \frac{1}{n}$ be the two numbers.

Clearly, $0 < |x_1 - 2| < \delta$ and $0 < |x_2 - 2| < \delta$

But $|f(x_1) - f(x_2)| = 2n \geq 2 > \epsilon$

$\therefore$ By Cauchy's Criterion, $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

Self Check Exercise: Show that $\lim_{x \to a}(x - a) \cos \frac{1}{x-a} = 0$.

Solution : ....

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1.2.5 Algebra of Limits

Let f, g be two real valued functions such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then the following results hold:

(i) $\lim_{x \to a}(f \pm g)(x) = l \pm m$,

(ii) $\lim_{x \to a}(fg(x)) = lm$,

(iii) $\lim_{x \to a} \frac{f}{g}(x) = \frac{l}{m}$, provided $m \neq 0$,

(iv) $\lim_{x \to a}(kf)(x) = kl$, where k is a real number,

(v) $\lim_{x \to a} |f(x)| = |l|$.

1.2.6 Squeeze Principle

Statement: If $f(x) \leq g(x) \leq h(x) \forall x$ in some deleted neighbourhood of a and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.

Proof: It is given that $\lim_{x\to a} f(x) = l$,

$\therefore$ for given $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$|f(x) - l| < \epsilon$ for $0 < |x - a| < \delta_1$

or $l - \epsilon < f(x) < l + \epsilon$ for $0 < |x - a| < \delta_1 \cdots (1)$

Again, $\lim_{x\to a} h(x) = l$,

$\therefore$ for given $\epsilon > 0$, there exists $\delta_2 > 0$ such that

or $l - \epsilon < f(x) < l + \epsilon$ for $0 < |x - a| < \delta_1 \cdots (2)$

Also $f(x) \leq g(x) \leq h(x) \forall x$ in some deleted neighbourhood of $a \cdots (3)$

Thus there exists some $\delta \leq min.\{\delta_1, \delta_2\}$ such that $(1), (2) and (3)$ hold in $0 < |x - a| < \delta$.

Example 3: Evaluate $\lim_{x\to 1} \frac{x^n-1}{x-1}$.

Solution: $\lim_{x\to 1} \frac{x^n-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^{n-1}+x^{n-2}+........+x+1)}{x-1}$

$= \lim_{x\to 1} x^{n-1} + x^{n-2} + ... + x + 1$

$= 1 + 1 + ... + 1 + 1(\text{ n times}) = n$

Example 4: Prove that $\lim_{x\to 0} \frac{x}{|x|+x^2}$ does not exist.

Solution: To evaluate $\lim_{x\to 0^-} \frac{x}{|x|+x^2}$, Put $x = 0 - h$ so that $h \to 0$ as $x \to 0^-$.

$\therefore \lim_{x\to 0^-} \frac{x}{|x|+x^2} = \lim_{h\to 0} \frac{0-h}{|0-h|+(0-h)^2} = lim_{h\to 0}\frac{-1}{1+h} = -1$

Again, to evaluate $\lim_{x\to 0^+} \frac{x}{|x|+x^2}$, Put $x = 0 + h$ so that $h \to 0$ as $x \to 0^+$.

$\therefore \lim_{x\to 0^+} \frac{x}{|x|+x^2} = \lim_{h\to 0} \frac{0+h}{|0+h|+(0+h)^2} = lim_{h\to 0}\frac{1}{1+h} = 1$

Clearly, $\lim_{x\to 0^-} \frac{x}{|x|+x^2} \neq \lim_{x\to 0^+} \frac{x}{|x|+x^2}$

$\therefore$, $\lim_{x\to 0} \frac{x}{|x|+x^2}$ does not exist .

Self Check Exercise: Show that $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

Solution : ....

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1.2.7 Summary : Following concepts are summarized from this lesson:

- If $a, b \in \mathbb{R}$ and $a < b$, then the set $\{x \in \mathbb{R} : a < x < b\}$ is called an open interval (a, b).
- Any open interval containing $c \in \mathbb{R}$ is known as the neighbourhood of point c.
- If $a < c < b$ i.e. $c \in (a, b)$, then $(a, c) \cup (c, b)$ is called the deleted neighbourhood of c.
- Limit of a function may exist at the point a even if the function is not defined at a.
- Limit of a function exists iff both left hand limit and right hand limit exist and are equal.
- Limit of a functions (if exists) is unique always.
- Cauchy's Criterion: If $\lim_{x \to a} f(x)$ exists, then for given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $0 < |x_1 - a| < \delta$ and $0 < |x_2 - a| < \delta$.

1.2.8 Glossary : Neighbourhood of a Point; Limit of a Function; Cauchy's Criterion for Limits; Algebra of Limits; Squeeze Principle.

1.2.9 Long Questions

1. State and prove Cauchy's Criterion for Limits.
2. State and prove Squeeze Principle.

3. Prove that $lim_{x \to \frac{1}{3}} x[\frac{1}{x}]$ does not exist.

1.2.10 Short Questions

1. Use definition of limit to prove that $\lim_{x \to 3}(11 - 2x) = 5$.
2. Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.
3. Evaluate $lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x^2}$

1.2.11 Suggested Readings

1. Shanti Narayan and P.K. Mittal, Differential Calculus, S. Chand and Co., New Delhi (Edition 2006).
2. Gorakh Prasad, Differential Calculus, Pothishala Pvt. Ltd. Allahabad.

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